THE DYING RABBIT PROBLEM

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1. INTRODUCTION

Fibonacci numbers originally arose in the answer to the following problem posed by Leonardo de Pisa in 1202. Suppose there is one pair of rabbits in an enclosure at the 0^{th} month, and that this pair breeds another pair in each of the succeeding months. Also suppose that pairs of rabbits breed in the second month following birth, and thereafter produce one pair monthly. What is the number of pairs of rabbits at the end of the n^{th} month? It is not difficult to establish by induction that the answer is F_{n+2}, where F_n is the n^{th} Fibonacci number. In [1] Brother Alfred asked for a solution to this problem if, like Socrates, our rabbits are motral, say each pair dies one year after birth. His answer [2], however, contained an error. The mistake was noted by Cohn [3], who also supplied the correct solution. In this paper we generalize the dying rabbit problem to arbitrary breeding patterns and death times.

2. SOLUTION TO THE GENERALIZED DYING RABBIT PROBLEM

Suppose that there is one pair of rabbits at the 0^{th} time point, that this pair produces B_1 pairs at the first time point, B_2 pairs at the second time point, and so forth, and that each offspring pair breeds in the same manner. We shall let $B_0 = 0$, and put

$$B(x) = \sum_{n=0}^{\infty} B_n x^n ,$$

so that $B(x)$ is the birth polynomial associated with the birth sequence

$$\{B_n\}_{n=0}^{\infty} .$$

The degree of $B(x)$, $\deg B(x)$, may be finite or infinite. Now suppose a pair of rabbits dies at the m^{th} time point after birth (after possible breeding), and let $D(x) = x^m$ be the associated death polynomial. If our rabbits are immortal, put $D(x) = 0$. Clearly $\deg D(x) > 0$ implies $\deg D(x) \geq \deg B(x)$, unless the rabbits have strange mating habits. Let T_n be the total number of live pairs of rabbits at the n^{th} time point, and put

$$T(x) = \sum_{n=0}^{\infty} T_n x^n ,$$

where $T_0 = 1$. Our problem is then to determine $T(x)$, where $B(x)$ and $D(x)$ are known.

Let R_n be the number of pairs of rabbits born at the n^{th} time point assuming no deaths. With the convention that the original pair was born at the 0^{th} time point, and recalling that $B_0 = 0$, we have

$$\begin{aligned} R_0 &= 1, \\ R_1 &= B_0R_1 + B_1R_0 , \\ R_2 &= B_0R_2 + B_1R_1 + B_2R_0 . \end{aligned}$$

and in general that

$$R_n = \sum_{j=0}^{n} B_j R_{n-j} \qquad (n \geq 1) . \tag{1}$$

Note that for $n = 0$ this expression yields the incorrect $R_0 = 0$. Then if

$$R(x) = \sum_{n=0}^{\infty} R_n x^n ,$$

equation (1) is equivalent to

$$R(x) = R(x)B(x) + 1 ,$$

so that

$$R(x) = \frac{1}{1 - B(x)} .$$

The total number T_n^* of pairs at the n^{th} time point assuming no deaths is given by

$$T_n^* = \sum_{j=0}^{n} R_j \;,$$

and we find

$$(2)\qquad \begin{aligned} \frac{1}{(1-x)[1-B(x)]} &= \frac{R(x)}{1-x} = \left(\sum_{k=0}^{\infty} x^k\right)\left(\sum_{n=0}^{\infty} R_n x^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} R_j\right)x^n = \sum_{n=0}^{\infty} T_n^* x^n = T^*(x)\;. \end{aligned}$$

Hoggatt [4] used slightly different methods to show both (1) and (2).

We must now allow for deaths. Since each pair dies m time points after birth, the number of deaths D_n at the n^{th} time point equals the number of births R_{n-m} at the $(n-m)^{th}$ time point. Therefore

$$\sum_{n=0}^{\infty} D_n x^n = D(x) \sum_{n=0}^{\infty} R_n x^n = \frac{D(x)}{1-B(x)}\;.$$

Letting the total number of dead pairs of rabbits at the n^{th} time point be

$$C_n = \sum_{j=0}^{n} D_j \;,$$

we have

$$\begin{aligned} \frac{D(x)}{(1-x)[1-B(x)]} &= \left(\sum_{k=0}^{\infty} x^k\right)\left(\sum_{n=0}^{\infty} D_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} D_j\right)x^n \\ &= \sum_{n=0}^{\infty} C_n x^n = C(x)\;. \end{aligned}$$

Now the total number of live pairs of rabbits T_n at the n^{th} time point is $T_n^* - C_n$, so that

$$(3) \qquad T(x) = T^*(x) - C(x) = \frac{1 - D(x)}{(1 - x)[1 - B(x)]}$$

3. SOME PARTICULAR CASES

To solve Brother Alfred's problem, we put $B(x) = x^2 + x^3 + \cdots + x^{12}$ and $D(x) = x^{12}$ in (3) to give

$$T(x) = \frac{1 - x^{12}}{(1 - x)(1 - x^2 - x^3 - \cdots - x^{12})} = \frac{1 - x^{12}}{1 - x - x^2 + x^{13}} .$$

It follows that the sequence $\{T_n\}$ obeys

$$T_{n+13} = T_{n+12} + T_{n+11} - T_n \qquad (n \geq 0) ,$$

together with the initial conditions $T_n = F_{n+1}$ for $n = 0, 1, \cdots, 11$, and $T_{12} = F_{13} - 1$, which agrees with the answer given by Cohn [3].

As another example of (3), suppose each pair produce a pair at each of the two time points following birth, and then die at the m^{th} time point after birth ($m \geq 2$). In this case, $B(x) = x + x^2$ and $D(x) = x^m$. From (3), we see

$$T(x) = \frac{1 - x^m}{(1 - x)(1 - x - x^2)} .$$

Making use of the generating function

$$\frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1}x^n ,$$

we get

$$T(x) = \frac{1 + x + \cdots + x^{m-1}}{1 - x - x^2} = \sum_{j=0}^{m-1} \frac{x^j}{1 - x - x^2}$$

$$(4) \quad = \sum_{j=0}^{m-1}\left(\sum_{n=0}^{\infty} F_{n+1}x^{n+j}\right) = \sum_{n=0}^{m-1}\left(\sum_{k=0}^{n} F_{k+1}\right)x^n + \sum_{n=m}^{\infty}\left(\sum_{k=0}^{m-1} F_{n-k+1}\right)x^n$$

$$= \sum_{n=0}^{m-1} (F_{n+3} - 1)x^n + \sum_{n=m}^{\infty} (F_{n+3} - F_{n-m+3})x^n .$$

For $m = 4r$ it is known [5] that

$$F_{n+3} - F_{n-4r+3} = F_{2r}L_{n-2r+3} ,$$

where L_n is the n^{th} Lucas number, while for $m = 4r + 2$,

$$F_{n+3} - F_{n-4r+1} = L_{2r+1}F_{n-2r+2} ,$$

which may be used to further simplify (4). In particular, for $m = 2$,

$$T(x) = 1 + 2x + \sum_{n=0}^{\infty} F_{n+2}x^n = \sum_{n=0}^{\infty} F_{n+2}x^n ,$$

while for $m = 4$ we have

$$T(x) = 1 + 2x + 4x^2 + 7x^3 + \sum_{n=4}^{\infty} L_{n+1}x^n$$

$$= -x + \sum_{n=0}^{\infty} L_{n+1}x^n .$$

Thus for proper choices of $B(x)$ and $D(x)$ we are able to get both Fibonacci and Lucas numbers as the total population numbers.

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