A FIBONACCI MATRIX AND THE PERMANENT FUNCTION

BRUCE W. KING
Burnt Hills-Ballston Lake High School, Ballston Lake, New York
and
FRANCIS D. PARKER
The St. Lawrence University, Canton, New York

The permanent of an n-square matrix $[a_{ij}]$ is defined to be

$$\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{ij_i} \quad ,$$

where

$$\sigma = (j_1, j_2, \cdots, j_n)^{\star}$$

is a member of the symmetric group S_n of permutations on n distinct objects. For example, the permanent of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is

$$a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31} \, .$$

This is similar to the definition of the determinant of $[a_{ij}]$, which is

$$\sum_{\sigma \in S_n} \epsilon_\sigma \prod_{i=1}^{n} a_{ij_i} \quad ,$$

where ϵ_σ is 1 or -1 depending upon whether σ is an even or an odd permutation.

There are other similarities between the permanent and the determinant functions, among them:

(a) interchanging two rows, or two columns, of a matrix changes the sign of the determinant — but it does not change the permanent at all. Thus, the permanent of a matrix remains invariant under arbitrary permutations of its rows and columns; and

*In this notation, $(j_1, j_2, \cdots, j_n)$ is an abbreviation for the permutation $\begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$.

(b) there is a Laplace expansion for the permanent of a matrix as well as for the determinant. In particular, there is a row or column expansion for the permanent. For example, if we use "per $[a_{ij}]$" for the permanent of the matrix $[a_{ij}]$, then expansion along the first column yields that

$$\text{per}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\,\text{per}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + a_{21}\,\text{per}\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{31}\,\text{per}\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}.$$

For further information on properties of the permanent, the reader should see [1, p. 578] and [3, pp. 25-26].

Unfortunately, one of the most useful properties of the determinant — its invariance under the addition of a multiple of a row (or column) to another row (or column) — is false for the permanent function. As a result, evaluating the permanent of a matrix is, generally, a much more difficult problem than evaluating the corresponding determinant.

Let P_n be the n-square matrix whose entries are all 0, except that each entry along the first diagonal above the main diagonal is equal to 1, and the entry in the n^{th} row and first column also is 1. (P_n is a "permutation matrix.") For example,

$$P_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The reader can verify that

$$P_5^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$P_5^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

We now define the matrix $Q(n,r)$ to be

$$\sum_{j=1}^{r} P_n^j .$$

For example,

$$Q(5,2) = P_5 + P_5^2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} .$$

It is not difficult to see that $\operatorname{per} Q(n,1) = \operatorname{per} P_n = 1$, $\operatorname{per} Q(n,2) = 2$, and that $\operatorname{per} Q(n,n) = n!$. It has been shown [2] that

$$\text{(1)} \qquad \operatorname{per} Q(n,3) = 2 + \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n .$$

The strategy used in the derivation of (1) was to use techniques for the solution of a linear difference equation on a certain recurrence involving $\operatorname{per} Q(n,3)$. There are, also, expressions available for $\operatorname{per} Q(n,4)$, $\operatorname{per} Q(n,n-1)$ and $\operatorname{per} Q(n,n-2)$. (See [3], [3, pp. 22-28] and [3, pp. 31-35], respectively.) However, $\operatorname{per} Q(n,r)$ has not been determined for $5 \le r \le n-3$. The objectives of this paper are to use a "Fibonacci matrix" to derive (1), and to derive an explicit expression for $\operatorname{per} Q(n,3)$ other than that provided by (1). (By a "Fibonacci matrix" we mean a matrix M_n for which $M_n = \operatorname{per} M_{n-1} + \operatorname{per} M_{n-2}$.)

Let F_n be the matrix $[f_{ij}]$, where $f_{ij} = 1$ if $|i-j| \le 1$ and $f_{ij} = 0$ otherwise. Then, by starting with an expansion along the first column, we find that F_n is a Fibonacci matrix.* Since $\operatorname{per} F_2 = 2$ and $\operatorname{per} F_3 = 3$, $\operatorname{per} F_n$ yields the $(n+1)^{\text{th}}$ term of the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$. It is well known that the n^{th} Fibonacci number is given by

*There are other Fibonacci matrices. See problem E1553 in the 1962 volume of the American Mathematical Monthly, for example.

$$\frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n\sqrt{5}} .$$

This, it follows that

$$\text{per } F_n = \frac{(1 + \sqrt{5})^{n+1} - (1 - \sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} . \tag{2}$$

It is not quite as well known that the n^{th} Fibonacci number is also given by

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n - k - 1}{k} ,$$

where

$$\left[\frac{n - 1}{2}\right]$$

is the greatest integer in

$$\frac{n - 1}{2} .$$

(See [4, pp. 13-14] for a proof.) From this it follows that

$$\text{per } F_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n - k}{k} . \tag{3}$$

Now let $U_n(i, j)$ be the n-square matrix all of whose entries are 0 except the entry in row i and column j, which is 1. If we let $R_n = F_n + U_n(n, 1)$, by expansion along the first row of R we find that

$$\text{per } R_n = \text{per } F_{n-1} + \text{per } [F_{n-1} - U_{n-1}(2,1) + U_{n-1}(n-1,1)] .$$

But, by expanding along the first column,

$$\text{per } [F_{n-1} - U_{n-1}(2,1) + U_{n-1}(n-1,1)] = \text{per } F_{n-2} + 1 . \tag{4}$$

Thus,

$$\text{per } R_n = \text{per } F_{n-1} + \text{per } F_{n-2} + 1 = 1 + \text{per } F_n .$$

If we now let $S_n = R_n + U_n(1,n)$, by expansion along the first row of S_n we find that

$$\begin{aligned} \text{per } S_n = \text{per } F_{n-1} + \text{per } &[F_{n-1} - U_{n-1}(2,1) + U_{n-1}(n-1,1)] \\ &+ \text{per } [Q(n-1,2) - U_{n-1}(n-2,1) \\ &- U_{n-1}(n-1,2) + P_{n-1}^{n-1}]. \end{aligned} \tag{5}$$

If we substitute from (4) and use Z for the matrix of the third term of the right member of (5), we have

$$\begin{aligned} \text{per } S_n &= \text{per } F_{n-1} + \text{per } F_{n-2} + 1 + \text{per } Z \\ &= \text{per } F_n + 1 + \text{per } Z. \end{aligned}$$

Now expand Z along its first column to obtain $\text{per } Z = 1 + \text{per } F_{n-2}$. Then

$$\text{per } S_n = 2 + \text{per } F_n + \text{per } F_{n-2} .$$

Since $\text{per } S_n = \text{per } Q(n,3)$ (because S_n can be obtained from $Q(n,3)$ by a permutation of columns), it follows that

$$\text{per } Q(n,3) = 2 + \text{per } F_n + \text{per } F_{n-2} .$$

By using (2), we obtain an expression for per $Q(n,3)$ which reduces to that given by Minc in [1]. By using (3), we obtain:

$$\operatorname{per} Q(n,3) = 2 + \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} + \sum_{k=0}^{\left[\frac{n-2}{2}\right]} \binom{n-k-2}{k} .$$

REFERENCES

1. Marvin Marcus and Henryk Minc, "Permanents," The American Mathematical Monthly, Vol. 72 (1965), No. 6, pp. 477-591.
2. Henryk Minc, "Permanents of (0,1)-Circulants," Canadian Mathematical Bulletin, Vol. 7 (1964), pp. 253-263.
3. Herbert J. Ryser, Combinatorial Mathematics, MAA Carus Monograph No. 14, J. Wiley and Sons, New York, 1963.
4. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell Publishing Company, New York, 1961.

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[Continued from page 538.]

SOLUTIONS TO PROBLEMS

1. $T_{n+1} = 5T_n + 2T_{n-1} - 9T_{n-2} - 5T_{n-3}$.

2. $T_{n+1} = 5T_n - 4T_{n-1} - 9T_{n-2} + 7T_{n-3} + 6T_{n-4}$.

3. $T_{n+1} = 5T_n - 7T_{n-1} + 3T_{n-2}$.

4. $T_{n+4} = 4T_{n+3} - 2T_{n+2} - 5T_{n+1} + 2T_n$.

5. $T_{n+6} = 2T_{n+5} + 4T_{n+4} - 4T_{n+3} - 6T_{n+2} + T_n$.

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