REMARK ON A PAPER BY R. L. DUNCAN CONCERNING THE UNIFORM DISTRIBUTION MOD 1 OF THE SEQUENCE OF THE LOGARITHMS OF THE FIBONACCI NUMBERS

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In the following we present a short proof of a theorem shown by R. L. Duncan [1]:

Theorem 1. If $\mu_1, \mu_2, \cdots$ is the sequence of the Fibonacci numbers, then the sequence $\log\mu_1, \log\mu_2, \cdots$ is uniformly distributed mod 1.

Moreover, we show the following proposition.

Theorem 2. The sequence of the integral parts $[\log \mu_1]$, $[\log \mu_2], \cdots$ of the logarithms of the Fibonacci numbers is uniformly distributed mod m for every positive integer $m \geq 2$.

Proof of Theorem 1. It is well known that

$$\frac{\mu_{n+1}}{\mu_n} \to \frac{1 + \sqrt{5}}{2} \quad ,$$

or

$$\log \mu_{n+1} - \log \mu_n \to \log \frac{1 + \sqrt{5}}{2} \quad , \quad \text{as } n \to \infty \; . \tag{1}$$

In [2] (see th. 12.2.1), it is shown that if $\omega \neq 0$ is real and algebraic, then θ^ω is not an algebraic number. Therefore,

$$\frac{1 + \sqrt{5}}{2}$$

being an algebraic number, we conclude that

$$\log \frac{1 + \sqrt{5}}{2}$$

is transcendental. (One can also argue as follows: let be given that $\theta > 0$ is algebraic. Now suppose that $\log \theta = u/v$ where u and v are integers. Then we would have $\theta^v = e^u$. But this is impossible since θ^v is algebraic and e^u is transcendental (orally communicated by A. M. Mark).

According to a theorem due to J. G. van der Corput we have that a sequence of real numbers $\lambda_1, \lambda_2, \cdots$ is uniformly distributed mod 1 if

$$\lambda_{n+1} - \lambda_n \longrightarrow \theta \quad \text{(an irrational number)} \ \text{as} \ n \to \infty .$$

(see [3]). By the property (1) we see that the sequence $\log \mu_1, \log \mu_2, \cdots$ is uniformly distributed mod 1.

Proof of Theorem 2. First, we use the fact that the sequence

$$\frac{\log \mu_n}{m} \quad (m, \ \text{an integer} \neq 0), \quad n = 1, 2, \cdots ,$$

is uniformly distributed mod 1 which follows by the same argument used in the proof of Theorem 1: we have namely

$$\frac{\log \mu_{n+1}}{m} - \frac{\log \mu_n}{m} \to \frac{\log \frac{1+\sqrt{5}}{2}}{m} \quad \text{(non-algebraic) as } n \to \infty.$$

Then according to a theorem of G. L. van den Eynden [4], quoted in [5] the sequence

$$[\log \mu_1], \ [\log \mu_2], \ \cdots$$

is uniformly distributed modulo m for every integer $m \geq 2$, that is, if $A(N, j, m)$ is the number of elements of the set

$$\{[\log \mu_n]\} \quad (n = 1, 2, \cdots, N) ,$$

satisfying

$$[\log \mu_n] \equiv j \ (\text{mod } m), \quad (0 \leq j \leq m - 1) ,$$

then

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