ON FIBONACCI AND LUCAS NUMBERS WHICH ARE PERFECT POWERS

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The Fibonacci numbers, defined for all rational integers n by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n ,$$

have for several centuries engaged the attention of mathematicians, and while many of their properties may be established by very simple methods, there are many unsolved problems connected with them. One such problem is to determine which Fibonacci numbers are perfect powers. The case of the Fibonacci squares was solved by J. H. E. Cohn in [3] and also in [4]. (See [5] for some applications of Cohn's method to other Diophantine problems.) Cohn showed that the only squares in the sequence F_n are given by

$$F_{-1} = F_1 = F_2 = 1, \quad F_0 = 0 \text{ and } F_{12} = 144 .$$

Having solved the problem of the Fibonacci squares, one is led to inquire as to which numbers F_n can be perfect cubes, fifth powers, etc. A proof that

$$F_1 = F_2 = 1, \quad F_6 = 8 \text{ and } F_{12} = 144$$

are the only perfect powers in the sequence F_n for positive n was given by Buchanan [1], but, unfortunately, Buchanan's proof was incomplete and was later retracted by him [2]. Thus the problem of determining all the perfect powers in the sequence F_n remains unsolved. In the present paper we first present a general criterion for solving this problem. We then apply our result to the case of the Fibonacci cubes and give the complete solution for this case. Finally, we give a similar criterion for determining which Lucas numbers are perfect powers, and determine all Lucas numbers which are perfect cubes.

To determine which numbers F_n are perfect k^{th} powers, we may assume, by Cohn's result, that $k = p$, where p is an odd prime, and also that n is positive, since $F_0 = 0$ and $F_{-n} = (-1)^{n+1}F_n$. Let L_n be the n^{th} term in the Lucas sequence, defined by

$$L_1 = 1, \quad L_2 = 3, \quad L_{n+2} = L_{n+1} + L_n ,$$

and let

$$a = \frac{1 + \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2} .$$

By induction, it is easily verified that

$$F_n = \frac{a^n - b^n}{\sqrt{5}}, \quad L_n = a^n + b^n ,$$

and since $ab = -1$, we have finally that

$$L_n^2 - 5F_n^2 = 4(-1)^n . \tag{1}$$

Let us first assume that n is even and that $F_n = z^p$, $L_n = y$, where p is an odd prime. Then (1) becomes

$$y^2 - 5z^{2p} = 4 . \tag{2}$$

Now it is clear that the solution of (2) reduces to the solution of

$$y^2 - 5x^p = 4 , \tag{3}$$

subject to

$$x = z^2 . \tag{4}$$

In (3) we set

$$x = \frac{X}{5}, \quad y = \frac{Y}{5^{(p-1)/2}} ,$$

which yields

$$Y^2 - 4 \cdot 5^{p-1} = X^p ,$$

subject to

(5) $$\frac{X}{5} = z^2, \quad X > 0, \quad Y > 0, \quad X \equiv 0 \pmod 5, \quad Y \equiv 0 \pmod{5^{(p-1)/2}} .$$

Similarly, if n is odd, the problem reduces to solving

$$Y^2 + 4 \cdot 5^{p-1} = X^p ,$$

subject to (5), and we have proved

Theorem 1. The problem of determining which numbers F_n, $n > 0$, are perfect p^{th} powers, where p is an odd prime, reduces to the solution of the equations

$$Y^2 + 4 \cdot 5^{p-1}(-1)^{n-1} = X^p ,$$

subject to the conditions

$$\frac{X}{5} = z^2, \quad X > 0, \quad Y > 0, \quad X \equiv 0 \pmod 5, \quad Y \equiv 0 \pmod{5^{(p-1)/2}} .$$

Let us apply Theorem 1 to the case $p = 3$. Here the problem reduces to solving

(6) $$Y^2 - 100 = X^3 ,$$

and

(7) $$Y^2 + 100 = X^3 ,$$

subject to the conditions

(8) $$\frac{X}{5} = z^2, \quad X > 0, \quad Y > 0, \quad X \equiv Y \equiv 0 \pmod 5 .$$

Now Hemer proved [7], [8], that the only solutions of (6) with $Y > 0$ are

$$[X,Y] = [-4,6],\quad [0,10],\quad [5,15],\quad [20,90],\ [24,118] \text{ and } [2660,137190].$$

Of these solutions, the only ones satisfying (8) are $[5,15]$ and $[20,90]$. This yields $[x,y] = [1,3]$ and $[4,18]$ as the only solutions of (3) (with $p = 3$) which satisfy (4), and from these solutions we derive

$$L_2 = 3,\quad F_2 = 1 \text{ and } L_6 = 18,\quad F_6 = 8 .$$

Thus the only cubes in the sequence F_n with n positive and even are $F_2 = 1$ and $F_6 = 8$.

In two previous papers [6], [10], we showed that the only integer solutions of (7) with $Y > 0$ are $[X,Y] = [5,5]$, $[10,30]$ and $[34,198]$. Of these solutions only $[5,5]$ satisfies (8), and from this solution we derive $F_1 = L_1 = 1$. Thus the only cube in the sequence F_n with n positive and odd is $F_1 = 1$, and we have

<u>Theorem 2.</u> The only cubes in the Fibonacci sequence F_n are

$$F_{-6} = -8,\ F_{-2} = -1,\ F_0 = 0,\ F_{-1} = F_1 = F_2 = 1 \text{ and } F_6 = 8 .$$

Next, we give a criterion for determining which Lucas numbers are perfect p^{th} powers, where p is an odd prime. We note that the case of the Lucas squares was solved by Cohn [3], who showed that the only Lucas squares are $L_1 = 1$ and $L_3 = 4$.

In (1) let $F_n = z$, $L_n = y^p$ and $n > 0$, and assume first that n is even. Then we get

$$y^{2p} - 5z^2 = 4 . \tag{9}$$

It is clear that (9) reduces to solving

$$x^p - 5z^2 = 4 , \tag{10}$$

subject to $x = y^2$.

Equation (10) may be written

$$5x^p - (5z)^2 = 20 \quad ,$$

and, setting $v = 5z$, it reduces to

$$5x^p - v^2 = 20 \, , \tag{11}$$

subject to

$$x = y^2, \quad v \equiv 0 \pmod 5 \, .$$

Finally, setting

$$x = \frac{X}{5}, \quad v = \frac{Y}{5^{(p-1)/2}} \quad ,$$

(11) reduces to

$$Y^2 + 4 \cdot 5^p = X^p \quad , \tag{12}$$

subject to the conditions

$$\frac{X}{5} = y^2, \quad X > 0, \quad Y > 0, \quad X \equiv 0 \pmod 5, \quad Y \equiv 0 \pmod{5^{(p+1)/2}}. \tag{13}$$

Similarly, if n is odd, the problem reduces to

$$Y^2 - 4 \cdot 5^p = X^p \quad ,$$

subject to the conditions (13), and we have

Theorem 3. The problem of determining all the perfect p^{th} powers in the sequence L_n, where p is an odd prime, reduces to solving the two equations

$$Y^2 + 4 \cdot 5^p(-1)^n = X^p \quad ,$$

subject to the conditions

$$\frac{X}{5} = y^2, \quad X > 0, \quad Y > 0, \quad X \equiv 0 \pmod 5, \quad Y \equiv 0 \pmod{5^{(p+1)/2}}.$$

Finally, we apply Theorem 3 to the case $p = 3$. Here the problem reduces to solving

$$Y^2 - 300 = X^3 , \tag{14}$$

and

$$Y^2 + 500 = X^3 , \tag{15}$$

subject to

$$\frac{X}{5} = y^2, \quad X > 0, \quad Y > 0, \quad X \equiv 0 \pmod 5, \quad Y \equiv 0 \pmod{25}. \tag{16}$$

In a previous paper [9], we showed that (15) is insoluble and that the only solution of (14) with $Y > 0$ is $[X,Y] = [5,25]$. This solution clearly fulfills (16) and also implies that $L_1 = F_1 = 1$. Thus we have proved

Theorem 4. The only cube in the Lucas sequence L_n, $n > 0$, is $L_1 = 1$.

In conclusion, we wish to point out that Siegel [11] has shown that the problem of determining all the complex quadratic fields of class number 1 can be reduced to the problem of finding all the cubes in the sequences F_n and L_n. Thus we have completed yet another proof of Gauss' famous conjecture on complex quadratic fields of class number 1.

REFERENCES

1. F. Buchanan, "N^{th} Powers in the Fibonacci Series," Am. Math. Monthly, 71 (1964), 647-649.
2. F. Buchanan, Retraction of "N^{th} Powers in the Fibonacci Series," Am. Math. Monthly, 71 (1964), 1112.

[Continued on page 487.]

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