LINEAR RECURSION RELATIONS — LESSON SEVEN
ANALYZING LINEAR RECURSION SEQUENCES

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Frequently one encounters problems such as the following: Find the next three terms in the following sequences:

$$1, \; 3, \; 5, \; 7, \; 9, \; 11, \; \cdots$$
$$3, \; 4, \; 7, \; 11, \; 18, \; 29, \; \cdots$$
$$3, \; 6, \; 12, \; 24, \; 48, \; 96, \; \cdots$$

As has been pointed out many times, the solution to such problems is highly indeterminate. It is "obvious" that the general term of the first sequence is

$$T_n = 2n - 1$$

But

$$T_n = 2n - 1 + (n - 1)(n - 2)(n - 3)(n - 4)(n - 5)(n - 6)V_n$$

where V_n is the n^{th} term of any sequence of finite quantities would do just as well. Similarly for the other cases.

Or looking at the matter from the standpoint of linear recursion relations, the six numbers in each case might be the first six terms of a linear recursion relation of the sixth order. Hence any infinite number of possibilities arises.

How can the problem be made more specific? Possibly, one might say: Find the expression for the n^{th} term of a linear recursion relation of minimum order. Whether this is sufficient to handle all instances of this type is an open question, but it would seem to take care of the present cases.

The solutions in the three instances listed above are:

$$T_{n+1} = 2T_n - T_{n-1}$$
$$T_{n+1} = T_n + T_{n-1}$$
$$T_{n+1} = 2T_n$$

If a sequence has terms which were derived from a polynomial expression in n, this expression can be found by the method of differences. As was pointed out in the first lesson, if the terms derive from a polynomial of degree k, the k^{th} differences are constant and the $(k+1)^{st}$ difference is zero. A simple method of reconstituting the polynomial is to use Newton's Interpolation Formula:

$$f(n) = \frac{\Delta^k f(0)}{k!} n^{(k)} + \frac{\Delta^{k-1} f(0) n^{(k-1)}}{(k-1)!} + \cdots + \frac{\Delta f(0) n^{(1)}}{1!} + f(0) \tag{1}$$

where $\Delta^k f(0)$ is the k^{th} difference taken at the zero value and $n^{(k)}$ is the factorial $n(n-1)(n-2)\cdots(n-k+1)$ of k terms.

Example. Determine the polynomial of lowest degree which fits the following set of values.

n	$f(n)$	$\Delta f(n)$	$\Delta^2 f(n)$	$\Delta^3 f(n)$
0	6			
		5		
1	11		32	
		37		18
2	48		50	
		87		18
3	135		68	
		155		18
4	290		86	
		241		18
5	531		104	
		345		18
6	876		122	
		467		18
7	1343		140	
		607		
8	1950			

Using Newton's Interpolation Formula,

$$f(n) = \frac{18}{3!} n(n-1)(n-2) + \frac{32}{2!} n(n-1) + 5n + 6$$

$$f(n) = 3n^3 + 7n^2 - 5n + 6.$$

Suppose that we have a sequence whose terms are the sum of the terms of two sequences: (1) A sequence whose values derive from a polynomial: (2) A sequence whose terms form a geometric progression. Is it possible to determine the components of this sequence?

Imagine that the terms of the sequence have been separated into their two component parts. Then on taking differences, the effect of the polynomial will eventually become nil. How does a geometric progression function under differencing? This can be seen from the table below.

a			
ar	$a(r-1)$	$a(r-1)^2$	
ar^2	$ar(r-1)$	$ar(r-1)^2$	$a(r-1)^3$
ar^3	$ar^2(r-1)$	$ar^2(r-1)^2$	$ar(r-1)^3$
ar^4	$ar^3(r-1)$	$ar^3(r-1)^2$	$ar^2(r-1)^3$

Clearly, differencing a geometric progression produces a geometric progression with the same ratio. By examining the form of the leading term, one can readily deduce the value of a, the initial term of the geometric progression as well.

<u>Example.</u>

POLYNOMIAL AND GEOMETRIC PROGRESSION COMBINED

n	T_n				
1	4				
		12			
2	16		42		
		54		58	
3	70		100		80
		154		138	
4	224		238		240
		392		378	
5	616		616		720
		1008		1098	
6	1624		1714		2160
		2722		3258	
7	4346		4972		6480
		7644		9738	
8	12040		14710		
		22404			
9	34444				

In the last column, one has a geometric progression with ratio 3, but not in the previous column. Hence the polynomial that was combined with the geometric progression was of degree 3. For the geometric progression, $r = 3$ and

$$a \times 2^4 = 80, \text{ so that } a = 5 .$$

Eliminating the effect of the geometric progression from the leading terms gives:

$$\begin{aligned} 58 - 2^3 \times 5 &= 18 \\ 42 - 2^2 \times 5 &= 22 \\ 12 - 2 \times 5 &= 2 \\ 4 - 5 &= -1 \end{aligned}$$

To apply Newton's Formula, we have to go back to the zero elements by extrapolation.

$$\Delta^3 f(0) = 18, \quad \Delta^2 f(0) = 22 - 18 = 4, \quad \Delta f(0) = 2 - 4 = -2$$
$$f(0) = -1 - (-2) = 1 .$$

Hence

$$f(n) = \frac{18}{3!} n(n - 1)(n - 2) + \frac{4}{2!} n(n - 1) - 2n + 1$$
$$f(n) = 3n^3 - 7n^2 + 2n + 1 .$$

Hence the term of the sequence has the form:

$$T_n = 3n^3 - 7n^2 + 2n + 1 + 5 \times 3^{n-1} .$$

The recursion relation for this term can be readily found by the methods of the previous lesson.

POLYNOMIAL AND FIBONACCI SEQUENCE

If we know that the terms of a sequence are formed by combining the elements of a polynomial sequence and a Fibonacci sequence, we have a situation similar to the previous case. For whereas the polynomial element vanishes on taking a sufficient number of differences, the Fibonacci element persists. This can be seen from the following table.

n	T_n	ΔT_n	$\Delta^2 T_n$	$\Delta^3 T_n$
1	T_1			
		T_0		
2	T_2			
		T_1		
3	T_3		T_{-1}	
		T_2		T_{-2}
4	T_4		T_0	
		T_3		T_{-1}
5	T_5		T_2	
		T_4		T_0
6	T_6		T_3	
		T_5		T_1
7	T_7		T_4	
		T_6		T_2
8	T_8			

<u>Example.</u>

n	T_n				
1	2				
		6			
2	8		21		
		27			
3	35		44	23	
		71			-6
4	106		61	17	
		132			5
5	238		83	22	
		215			-1
6	453		104	21	
		319			4
7	772		129	25	
		448			3
8	1220		157	28	
		605			
9	1825				

The last column has a Fibonacci property, but the previous column does not. Hence the polynomial must have been of degree three. We identify the first terms of the Fibonacci sequence as being 3, the zero term as 4, the term with -1 subscript as -1, etc. The effect of these terms can be eliminated from the leading edge of the table to give: 23 - 5 = 18; 21 - (-1) = 22; 6 - 4 = 2; 2 - 3 = -1. Calculating the zero differences as before, the final form of the term to be found is:

$$T_n = 3n^3 - 7n^2 + 2n + 1 + V_n$$

where

$$V_1 = 3, \quad V_2 = 7, \quad \text{and} \quad V_{n+1} = V_n + V_{n-1} .$$

PROBLEMS

1. Determine the polynomial for which $f(1) = -4$; $f(2) = 22$; $f(3) = 100$; $f(4) = 200$; $f(5) = 532$; $f(6) = 946$; $f(7) = 1532$; $f(8) = 2320$.

2. The following sequence of values correspond to terms T_1, T_2, etc. of a sequence which is the sum of a polynomial and a Fibonacci sequence: 0, 4, 12, 29, 53, 87, 132, 192, 272, 381. Determine the polynomial and the Fibonacci sequence components.

3. The values: 13, 72, 227, 526, 1023, 1784, 2899, 4506, 6839 include a polynomial component and a geometric progression component. Determine the general form of the term of the sequence.

4. The sequence values: 4, 14, 12, 22, 20, 30, 28, 38, 36, $\cdots$ combine a polynomial and a geometric progression. Determine the general form of the term of the sequence.

5. The sequence values: 7, 19, 45, 109, 219, 395, 653, 1017, 1515 have a polynomial and a Fibonacci component. Determine the general form of the polynomial and find the Fibonacci sequence.

(Solutions to these problems can be found on page 112.)

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CORRECTION

Please make the following changes to "Remark on a Theorem by Waksman," appearing in the Fibonacci Quarterly, October, 1969, p. 230.

On line 1, change "$Q = Q^* \cup \{1\}$" to "$Q^* = Q \cup \{1\}$"

On line 9, change "[2, p. 62]" to "[2, § 62]"

On line 18, change "$\notin V \cap Q$" to "$\in V \cap U$"

On line 20, change "a prime" to "an integer $p \in Q^*$."