AN EXTENSION OF THE FIBONACCI NUMBERS (PART II)

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In this section we consider the generalized Fibonacci and Tribonacci numbers.

We write the generalized Fibonacci numbers as

(1) $$(1 - a_1x - a_2x^2)^{-k} = \sum_{v=0}^{\infty} F_v^{(k)} x^v \quad (F_v = F_v^{(1)}) ,$$

where

$$F_n = a_1F_{n-1} + a_2F_{n-2}, \quad F_0 = 1, \quad F_1 = a_1, \quad F_2 = a_1^2 + a_2,$$
$$k = 1, 2, 3, \cdots \quad \text{and} \quad n = 0, 1, 2, \cdots .$$

The generalized Tribonacci numbers we write as

(2) $$(1 - a_1x - a_2x^2 - a_3x^3) = \sum_{v=0}^{\infty} T_v^{(k)} x^v ,$$

where

$$T_v = T_v^{(1)}, \quad T_0 = 1, \quad T_1 = a_1, \quad T_2 = a_1^2 + a_2, \quad F_3 = a_1^3 + 2a_1a_2 + a_3,$$
$$T_n = a_1T_{n-1} + T_{n-2}a_2 + a_3T_{n-3}, \quad k = 1,2,3,\cdots \quad \text{and} \quad n = 0, 1, 2,\cdots.$$

Note: Throughout this section we consider a_1, a_2, and a_3, as rational integers only.

CONVOLUTED SUM FORMULAS FOR THE GENERALIZED FIBONACCI AND TRIBONACCI NUMBERS

By elementary means, it is easy to prove, if

$$(1 - y)^{-k} = \sum_{v=0}^{\infty} b_v^{(k)} y^v \tag{3}$$

then

$$\binom{n + k - 1}{k - 1} = b_n^{(k)} ,$$

where

$$\binom{n + k - 1}{k - 1} = (n + k - 1)!/n!\,(k - 1)! , \quad b_0^{(k)} = 1, \quad k = 1, 2, 3, \cdots \text{ and } n = 0, 1, 2, \cdots .$$

Now, in (1), we replace $a_1x + a_2x^2$ with y so that combining (1) with (3) we may then write

$$\sum_{v=0}^{\infty} b_v^{(k)} y^v = \sum_{v=0}^{\infty} F_v^{(k)} x^v ,$$

It is easy to prove with induction that

$$\sum_{j=0} b_{n-j}^{(k)} \binom{n - j}{j} a_1^{n-2j} a_2^j = F_n^{(k)} ,$$

and combining this result with

$$b^{(k)} = \binom{n + k - 1}{k - 1}$$

leads to the following generalized Fibonacci convoluted sum formula:

$$F_n^{(k)} = \sum_{j=0} \binom{n + k - 1 - j}{k - 1} \binom{n - j}{j} a_1^{n-2j} a_2^j \qquad (4)$$

$$(n = 0,1,2,\cdots, k = 1,2,3,\cdots).$$

Now in (2), we replace $a_1x + a_2x^2 + a_3x^3$ with y so that combining (2) with (3), we may then write

$$\sum_{v=0}^{\infty} b_v^{(k)} y^v = \sum_{v=0}^{\infty} T_v^{(k)} x^v ,$$

and by comparing coefficients, it is easy to prove with induction, that

$$T_n^{(k)} = \sum_{r=0} \sum_{j=0}^{r} \left[b_{n-2r}^{(k)} \binom{n - 2r}{2r - j} \binom{2r - j}{j} a_1^{n-4r+j} a_2^{2r-2j} a_3^j + b_{n-2r-1}^{(k)} \binom{n - 2r - 1}{2r - 1 + j} \binom{2r + 1 - j}{j} a_1^{n-4r-2+j} a_2^{2r+1-2j} a_3^j \right]$$

and combining this result with

$$b_n^{(k)} = \binom{n + k - 1}{k - 1}$$

leads to the following generalized Tribonacci convoluted sum formula:

$$T_n^{(k)} = \sum_{r=0} \sum_{j=0}^{r} \left[\binom{k + n - 2r - 1}{k - 1} \binom{n - 2r}{2r - j} \binom{2r - j}{j} a_1^{n-4r+j} a_2^{2r-2j} a_3^j + \binom{k+n-2r-2}{k-1} \binom{n-2r-1}{2r+1-j} \binom{2r+1-j}{j} a_1^{n-4r-2+j} a_2^{2r+1-2j} a_3^j \right] \qquad (5)$$

where $n = 0, 1, 2, \cdots$ and $k = 1, 2, 3, \cdots$.

THE GENERALIZED FIBONACCI NUMBER EXPRESSED EXPLICITLY AS A DETERMINANT

We shall now prove the following five statements:

I. $$nF_n^{(k)} = a_1(k + n - 1)F_{n-1}^{(k)} + a_2(2k + n - 2)F_{n-2}^{(k)} ,$$

where

$$F_0^{(k)} = 1, \quad F_1^{(k)} = a_1k, \quad n = 2, 3, \cdots, \quad k = 2, 3, \cdots .$$

II. $$\frac{nF_n^{(k)}}{F_{n-1}^{(k)}} = p_1 + \frac{q_2}{p_2 +} \frac{q_3}{p_3 +} \cdots + \frac{q_{n-1}}{p_{n-1} +} \frac{q_n}{p_n}$$

where $p_j = a_1(k + n - j) \quad (j = 1, 2, 3, \cdots, n),$

$q_{m+1} = a_2(n - m)(2k + n - m - 1), \quad (m = 1, 2, 3, \cdots, n - 1),$

$(n = 2, 3, \cdots)$

$(k = 2, 3, \cdots)$,

$F_0^{(k)} = 1,$

$F_1^{(k)} = a_1k$.

III. $$(a_1^2 + 4a_2)kF_{n-1}^{(k+1)} = a_1nF_n^{(k)} + a_2(4k + 2n - 2)F_{n-1}^{(k)} ,$$

where $F_0^{(k)} = 1$

$F_1^{(k)} = a_1k$

$n = 1, 2, \cdots$

$k = 1, 2, 3, \cdots$.

IV. $$\sum_{v=1}^{n} F_{n-v}^{(v)} = \frac{((a_1+1+((a_1+1)^2+4a_2)^{\frac{1}{2}}+4a_2)^{\frac{1}{2}})/2)^n - ((a_1+1-((a_1+1)^2+4a_2)^{\frac{1}{2}})/2)^n}{((a_1+1)^2 + 4a_2)^{\frac{1}{2}}}$$

where $n = 1, 2, 3, \cdots$.

V. $F_n^{(k)} = K(p_1, q_2, \cdots, q_n, p_n)/n!$, ($p_n, q_n$ are identical to those in (ii) with $q_1 = 0$).

where $n,k = 1,2,3,\cdots$ and $K(p_1, q_2, \cdots, q_n + p_n)$ is the determinant given below in (6).

$$(6)\qquad K(p_1, q_2, \cdots, q_n, p_n) = \begin{bmatrix} p_1 & q_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & p_2 & q_3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & p_3 & q_4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & p_4 & q_5 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & p_{n-1} & q_n \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & p_n \end{bmatrix}$$

The table below of the generalized Fibonacci Numbers (in the table, we have replaced a_1, a_2 in (1) with $a_1 = a$ and $a_2 = b$)

(7)

	0	1	2	3	4	$\cdots$
0	0	0	0	0	0	$\cdots$
1	1	a	$a^2 + b$	$a^3 + 2ab$	$a^4 + 3a^2b + b^2$	$\cdots$
2	1	$2a$	$3a^2 + 2b$	$4a^3 + 6ab$	$5a^4 + 12a^2b + 3b^2$	$\cdots$
3	1	$3a$	$6a^2 + 3b$	$10a^3 + 12ab$	$15a^4 + 30a^2b + 6b^2$	$\cdots$
.	.	.	.	.	.	$\cdots$
.	.	.	.	.	.	$\cdots$
k	1	ka	.	.	.	$\cdots$
.	.	.	.	.	.	$\cdots$
.	.	.	.	.	.	$\cdots$

may be constructed as follows:

(8) To get the k^{th} element in the n^{th} column, we add the product of $\underline{a}$ multiplied by the k^{th} element in the $(n-1)^{st}$ column and the product of $\underline{b}$

multiplied by the k^{th} element in the $(n-2)^{nd}$ column together with the $(k-1)^{st}$ element in the n^{th} column.

We write the k^{th} element in the n^{th} column as $F_n^{(k)}$, so that a restatement of (8) reads

$$F_n^{(k)} = a_1F_{n-1}^{(k)} + a_2F_{n-2}^{(k)} + F_n^{(k-1)}, \tag{9}$$

where

$$\begin{aligned} F_0^{(k)} &= 1 \\ F_1^{(k)} &= a_1k \\ 0 &= F_0^{(0)} = F_1^{(0)} = F_2^{(0)} = \cdots \\ n &= 2, 3, \cdots \\ k &= 1,2,3,\cdots . \end{aligned}$$

PROOF OF I, II, III, AND IV

We use (9) to get

$$\begin{aligned} \sum_{n=2}^{\infty} F_n^{(k)} x^n &= a_1 \sum_{n=2}^{\infty} F_{n-1}^{(k)} x^n + a_2 \sum_{n=2}^{\infty} F_{n-2}^{(k)} x^n + \sum_{n=2}^{\infty} F_n^{(k-1)} x^n \\ &= a_1x \sum_{n=1}^{\infty} F_n^{(k)} x^n + a_2x^2 \sum_{n=0}^{\infty} F_n^{(k)} x^n + \sum_{n=2}^{\infty} F_n^{(k-1)} x^n , \end{aligned} \tag{10}$$

for $k = 1, 2, \cdots$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} F_n^{(k)} x^n - F_0^{(k)} - F_1^{(k)} x = a_1x \sum_{n=0}^{\infty} F_n^{(k)} x^n - a_1F_0^{(k)} x + a_2x^2 \sum_{n=0}^{\infty} F_n^{(k)} x^n \\ + \sum_{n=0}^{\infty} F_n^{(k-1)} x^n - F_0^{(k-1)} - F_1^{(k-1)}x , \end{aligned}$$

and therefore

$$(1 - a_1x - a_2x^2)\sum_{n=0}^{\infty} F_n^{(k)} x^n = F_0^{(k)} - F_0^{(k-1)} + (F_1^{(k)} - a_1F_0^{(k)} - F_1^{(k-1)})x + \sum_{n=0}^{\infty} F_n^{(k-1)} x^n .$$

Now

$$F_0^{(k)} - F_0^{(k-1)} = \begin{cases} 1 - 0 = 1 \text{ if } k = 1 \\ 1 - 1 = 0 \text{ if } k \quad 2 \end{cases}$$

and

$$F_1^{(k)} - a_1F_0^{(k)} - F_1^{(k-1)} = \begin{Bmatrix} a_1 - a_1 - 0 = 0 & \text{if } k = 1 \\ a_1k - a_1 - a_1(k - 1) = 0 & \text{if } k \quad 2 \end{Bmatrix} = 0 ,$$

for $k = 1, 2, 3, \cdots,$ and

$$\sum_{n=0}^{\infty} F_n^{(0)} = 0 .$$

Therefore

$$\sum_{n=0}^{\infty} F_n^{(k)} x^n = (1 - a_1x - a_2x^2)^{-1}\left(\sum_{n=0}^{\infty} F_n^{(k-1)} x^n\right) \qquad (k = 2, 3, \cdots),$$

and

$$\sum_{n=0}^{\infty} F_n^{(1)} x^n = (1 - a_1x - a_2x^2)^{-1} .$$

From this, we have at once

$$(11) \qquad (1 - a_1x - a_2x^2)^{-k} = \sum_{n=0}^{\infty} F_n^{(k)} x^n \qquad (k = 1,2,3,\cdots) .$$

Differentiation of (11) leads to

$$k(2a_2x + a_1)\left(\sum_{n=0}^{\infty} F_n^{(k+1)}x^n\right) = \sum_{n=1}^{\infty} nF_n^{(k)}x^{n-1} ,$$

and comparing the coefficients we conclude that

$$(12) \quad k(a_1F_{n-1}^{(k+1)} + 2a_2F_{n-2}^{(k+1)}) = nF_n^{(k)} \qquad (k = 1,2,3,\cdots, n = 2,3,\cdots).$$

Combining (12) with (9), we get

$$(13) \qquad nF_n^{(k)} = a_1(k + n - 1)F_{n-1}^{(k)} + a_2(2k + n - 2)F_{n-2}^{(k)}$$

for $k = 2,3,\cdots$, $n = 2,3,\cdots$, $F_0^{(k)} = 1$, and $F_1^{(k)} = a_1k$. This completes the proof for I.

When we divide (13) by $F_{n-1}^{(k)}$, we have

$$\frac{nF_n^{(k)}}{F_{n-1}^{(k)}} = a_1(k + n - 1) + \frac{a_2(2k + n - 2)(n - 1)}{\dfrac{(n - 1)F_{n-1}^{(k)}}{F_{n-2}^{(k)}}} \qquad (n=2,3,\cdots,k=2,3,\cdots)$$

which in turn along with $F_0^{(k)} = 1$ and $F_1^{(k)} = a_1k$, implies II.

The identity

$$a_1^2 + 4a_2 = 4a_2(1 - a_1x - a_2x^2) + (a_1 + 2a_2x)^2$$

may be written as

$$\frac{a_1^2 + 4a_2}{(1 - a_1x - a_2x^2)^k} = \frac{4a_2}{(1 - a_1x - a_2x^2)^{k-1}} + \frac{(a_1 + 2a_2x)^2}{(1 - a_1x - a_2x^2)^k} \tag{14}$$

$$(k = 1,2,3,\cdots) .$$

Differentiation leads to

$$\frac{(a_1^2 + 4a_2)kx}{(1 - a_1x - a_2x^2)^{k+1}} = \frac{4a_2kx}{(1 - a_1x - a_2x^2)^k} + (a_1 + 2a_2x)\left(\sum_{n=1}^{\infty} nF_n^{(k)} x^n\right).$$

Now, by comparing coefficients, we conclude that

$$(a_1^2 + 4a_2)kF_{n-1}^{(k+1)} = a_1nF_n^{(k)} + a_2(4k + 2n - 2)F_{n-1}^{(k)} , \tag{15}$$

when $F_0^{(k)} = 1$, $F_1^{(k)} = a_1k$, $n = 1,2,3,\cdots$, and $k = 1,2,3,\cdots$, which proves III.

We observe that Equations (II) and (III) immediately give an expression for

$$\frac{(a_1^2 + 4a_2)F_{n-1}^{(k+1)}}{F_{n-1}^{(k)}}$$

in the form of a continued fraction, for $n = 2, 3, \cdots$, and $k = 2, 3, \cdots$.

(Proof of IV). In (9), we have

$$F_n^{(k)} = a_1F_{n-1}^{(k)} + a_2F_{n-2}^{(k)} + F_n^{(k-1)} ,$$

so that

$$\sum_{v=1}^{n} F_{n-v}^{(v)} = a_1 \sum_{v=1}^{n-1} F_{n-v-1}^{(v)} + a_2 \sum_{v=1}^{n-2} F_{n-v-2}^{(v)} + \sum_{v=2}^{n} F_{n-v}^{(v-1)} \qquad (n=2,3,\cdots). \tag{16}$$

We see that

$$\sum_{v=1}^{n-1} F_{n-v-1}^{(v)} = \sum_{v=2}^{n} F_{n-v}^{(v-1)} ,$$

and we write (16) as

$$\sum_{v=1}^{n} F_{n-v}^{(v)} = (a_1 + 1)\left(\sum_{v=1}^{n-1} F_{n-v-1}^{(v)}\right) + a_2 \sum_{v=1}^{n-2} F_{n-v-2}^{(v)} . \tag{17}$$

We let

$$u_n = \sum_{v=1}^{n} F_{n-v}^{(v)} ;$$

then

$$u_{n-1} = \sum_{v=1}^{n-1} F_{n-v-1}^{(v)} , \qquad \text{and} \qquad u_{n-2} = \sum_{v=1}^{n-2} F_{n-v-2}^{(v)} ,$$

so that (17) becomes

$$u_n = (a_1 + 1)u_{n-1} + a_2 u_{n-2} . \tag{18}$$

Replacing n with $n + 2$ in (18), we have

$$u_{n+2} = (a_1 + 1)u_{n+1} + a_2 u_n , \tag{19}$$

where

$$u_1 = F_0 = 1, \ F_0^{(2)} + F_1 = 1 + a_1 = u_2, \quad \text{and} \quad n = 1, 2, 3, \cdots .$$

We now solve (19) for u_n by continued fractions (see [1]), and get

$$u_n = ((a_1 + 1 + s)^n - (a_1 + 1 - s)^n)/2^n s = \sum_{v=1}^{n} F_{n-v}^{(v)} ,$$

where

$$s = ((a_1 + 1)^2 + 4a_2)^{\frac{1}{2}}, \quad n = 1, 2, 3, \cdots ,$$

which completes the proof of IV.

PROOF OF V

Combining Euler's expression for a continuant as a determinant (see [2]) with (II) and (6), leads to

$$\text{(20)} \qquad \frac{nF_n^{(k)}}{F_{n-1}^{(k)}} = \frac{K(p_1, q_2, \cdots, q_n, p_n)}{K(p_2, q_3, \cdots, q_n, p_n)} ,$$

for $n,k = 2,3,4,\cdots$.

Note: For convenience we let

$$F_n^{(k)} / F_{n-1}^{(k)} = U_k(n) .$$

Now, using the values of p_j and q_{m+1} in (II), we write

$$\text{(21)} \qquad nU_k(n) = \frac{K(a_1(k + n - 1), a_2(n - 1)(2k + n - 2), \cdots, a_2(2k), a_1k)}{K(a_1(k + n - 2), a_1(n - 2)(2k + n - 3), \cdots, a_2(2k), a_1k)} ,$$

$$(n - 1)U_k(n - 1) = \frac{K(a_1(k + n - 2), a_2(n - 2)(2k + n - 3), \cdots, a_2(2k), a_1k)}{k(a_1(k + n - 3), a_2(n - 3)(2k + n - 4), \cdots, a_2(2k), a_1k)} ,$$

. . . .

$$3U_k(3) = \frac{K(a_1(k + 2), a_2(2)(2k + 1), \cdots, a_2(2k), a_1k)}{\begin{vmatrix} a_1(k + 1) & a_2(2k) \\ -1 & a_1k \end{vmatrix}}$$

$$2U_k(2) = \frac{\begin{vmatrix} a_1(k+1) & a_2(2k) \\ -1 & a_1k \end{vmatrix}}{a_1k} .$$

We now multiply all the equations in (21) from top to bottom to get

$$n! \prod_{j=2}^{n} U_k(j) = n!\, F_n^{(k)} / F_1^{(k)} = K(p_1, q_2, \cdots, q_n, p_n)/a_1k \quad , \tag{22}$$

for $n,k = 2, 3, 4, \cdots$.

Now combining (II, with $F_1^{(k)} = a_1k$) with (22) completes the proof of V.

We resolve for $k = 1$ $(n = 0, 1, 2, \cdots)$ by the use of continued fractions (see [1]), and we have

$$F_n = ((a_1 + V)^{n+1} - (a_1 - V)^{n+1})/V2^{n+1} ,$$

where

$$V = (a_1^2 + 4a_2)^{\frac{1}{2}} ,$$

and

$$F_n = a_1F_{n-1} + a_2F_{n-2} \; (F_0 = 1, \; F_1 = a_1) .$$

FORMULAS

For $F_n^{(t)}$ ($t = 2$, 3, and 4) as a function of F_{n-1} and F_n.

Let

$$A = a_1^2 + 4a_2, \; B(k,n) = 4k + 2n - 2 ,$$

where

$$F_0^{(k)} = 1, \quad F_1^{(k)} = a_1 k, \quad n,k = 1,2,3,\cdots,$$

and

$$F_n = a_1 F_{n-1} + a_2 F_{n-2},$$

(where a_1 and a_2 are rational integers); then from (III), we have

$$AkF_{n-1}^{(k+1)} = a_1 n F_n^{(k)} + a_2 B(k,n) F_{n-1}^{(k)} . \tag{23}$$

In (23), we have the following: when $k = 1$, then

$$AF_{n-1}^{(2)} = a_1 n F_n + a_2 B(1,n) F_{n-1},\tag{24}$$

when $k = 2$, then

$$2AF_{n-1}^{(3)} = a_1 n F_n^{(2)} + a_2 B(2,n) F_{n-1}^{(2)},$$

so that multiplying by $1{:}A$, we get

$$2!\, A^2 F_{n-1}^{(3)} = a_1 n A F_n^{(2)} + a_2 B(2,n) A F_{n-1}^{(2)} ,$$

and combining this with (24), we write (using the identity $F_n = a_1 F_{n-1} + a_2 F_{n-2}$)

$$\begin{aligned} 2!\, A^2 F_{n-1}^{(3)} &= a_2 B(2,n)(a_1 n F_n + a_2 B(1,n) F_{n-1}) \\ &\quad + a_1 n (a_1 (n+1) F_{n+1} + a_2 B(1, n+1) F_n) \end{aligned} \tag{25}$$

and replacing F_{n+1} (in (25)) with $F_{n+1} = a_1 F_n + a_2 F_{n-1}$ leads to

$$A^2F_{n-1}^{(3)} = \Big[(a_1a_2nB(1,n+1) + a_1a_2nB(2,n) + a_1^3n(n+1))F_n \tag{26}$$
$$+ (a_2^2B(1,n)B(2,n) + a_1^2a_2n(n+1))F_{n-1}\Big],$$

when $k = 3$, then in the exact way we found (26), we prove that

$$3!\,A^3F_{n-1}^{(4)} = M + N, \tag{27}$$

where

$$M = \begin{bmatrix} a_1a_2^2nB(1,n+1)B(3,n) + a_1a_2^2nB(2,n)B(3,n) \\ + a_1^3a_2n(n+1)B(3,n) + a_1a_2^2nB(1,n+1)B(2,n+1) \\ + a_1^3a_2n(n+1)(n+2) + a_1^3a_2n(n+1)B(1,n+2) \\ + a_1^3a_2n(n+1)B(2,n+1) + a_1^5n(n+1)(n+2) \end{bmatrix} F_n,$$

and

$$N = \begin{bmatrix} a_2^3B(1,n)B(2,n)B(3,n) + a_1^2a_2^2n(n+1)B(3,n) \\ + a_1^2a_2^2n(n+1)B(1,n+2) + a_1^2a_2^2n(n+1)B(2,m+1) \\ + a_1^4a_2n(n+1)(n+2) \end{bmatrix} F_{n-1}.$$

REMARKS

The above method may be used to evaluate formulas of the $F_n^{(k)}$ for values of $k = 5$ and higher.

THE GENERALIZED FIBONACCI NUMBER EXPRESSED AS A LIMIT

We now prove that

$$\text{VI} \qquad \lim_{n\to\infty} \left(F_n^{(k+1)}/(n+1)^k F_n\right) = \left(1 + a_1(a_1^2 + 4a_2)^{-\frac{1}{2}}\right)^k / 2^k k! \ ,$$

when

$$\lim_{n\to\infty} ((4k-2)/n) = 0 \qquad (k, n = 1, 2, 3, \cdots) .$$

Let

$$(28) \qquad A = a_1^2 + 4a_2, \quad V = A^{\frac{1}{2}}, \quad H = \tfrac{1}{2}(a_1 + V) \ ,$$

where

$$F_n = a_1 F_{n-1} + a_2 F_{n-2}, \quad F_0 = 1, \quad F_1 = a_1, \quad \text{and} \quad a_1, a_2$$

are rational integers.

It is easy to prove by use of continued fractions (see [1]) that

$$F_n = ((a_1 + V)^{n+1} - (a_1 - V)^{n+1})/2^{n+1} V \qquad (n = 0, 1, 2, \cdots) \ ,$$

and then by elementary means we show that

$$(29) \qquad \lim_{n\to\infty} (F_n/F_{n-1}) = \tfrac{1}{2}(a_1 + V) = H .$$

Now, combining (28) with (III), we have

$$(30) \qquad AkF_{n-1}^{(k+1)} = a_1 n F_n^{(k)} + a_2(4k + 2n - 2)F_{n-1}^{(k)} \ ,$$

where $n,k = 1, 2, 3, \cdots$.

In (30), we have the following: when $k = 1$, then

$$AF_{n-1}^{(2)} = a_1 nF_n + a_2(2n + 2)F_{n-1} ,$$

and dividing this equation by nF_{n-1}, we have

$$\frac{AF_{n-1}^{(2)}}{nF_{n-1}} = \frac{a_1F_n}{F_{n-1}} + a_2\left(\frac{2n + 2}{n}\right) ,$$

where combining this result with (29), we write

$$(31)\qquad A\left(\lim_{n\to\infty} F_{n-1}^{(2)}/nF_{n-1}\right) = \lim_{n\to\infty} \left(a_1F_n/F_{n-1} + a_2(2n+2)/n\right) = a_1H + 2a_2 ;$$

when $k = 2$ (in (30)), then

$$(32)\qquad 2AF_{n-1}^{(3)} = a_1 nF_n^{(2)}(F_n/F_n) + a_2(2n + 6)F_{n-1}^{(2)} ,$$

Multiplying both sides of (32) by A/n^2F_{n-1}, we now write

$$(33)\qquad \begin{aligned} 2A^2(F_{n-1}^{(3)}/n^2F_{n-1}) &= a_1(AF_n^{(2)}/F_n)(1/n)(F_n/F_{n-1}) \\ &\quad + a_2\left(\frac{2n + 6}{n}\right)(AF_{n-1}^{(2)}/nF_{n-1}) . \end{aligned}$$

Then combining (33) with (31) leads to

$$(34)\qquad \begin{aligned} 2A^2\left(\lim_{n\to\infty} F_{n-1}^{(3)}/n^2F_{n-1}\right) &= a_1(a_1H + 2a_2)H + 2a_2(a_1H + 2a_2) \\ &= (a_1H + 2a_2)^2 ; \end{aligned}$$

when $k = 3$ (in (30)), then

$$(35)\qquad 3AF_{n-1}^{(4)} = a_1 nF_n^{(3)}(F_n/F_n) + a_2(2n + 10)F_{n-1}^{(3)} ,$$

multiplying both sides of (35) by $2A^2/n^3F_{n-1}$, we now write

(36)
$$3!\,A^3(F_{n-1}^{(4)}/n^3F_{n-1}) = a_1(2A^2F_n^{(3)}/n^2F_n)(F_n/F_{n-1}) + a_2\left(\frac{2n+10}{n}\right)(2A^2F_{n-1}^{(3)}/n^2F_{n-1})$$

where combining (36) with (34) leads to

(37)
$$3!\,A^3(\lim_{n\to\infty} F_{n-1}^{(4)}/n^3F_{n-1}) = a_1(a_1H + 2a_2)^2H + 2a_2(a_1H + 2a_2)^2 = (a_1H + 2a_2)^3\,.$$

Then, step-by-step, and with induction, we prove that

(38)
$$k!\,A^k \lim_{n\to\infty}(F_n^{(k+1)}/(n+1)^kF_n) = (a_1H + 2a_2)^k\,,$$

where replacing the A and H in (38) with their respective values in (28), we complete the proof of VI.

REMARK. It may be interesting to note that if $a_1^2 + 4a_2$ is replaced by $a_1^2 + 4a_2 = (a_1k)^2$ in the right side of VI, then of course

$$\lim_{\substack{n\to\infty\\k\to\infty}}(2^kk!\,F_n^{(k+1)}/(n+1)^kF_n) = e \qquad (e = 2.71828\cdots)\,.$$

AN EXPLICIT FORMULA FOR THE TRIBONACCI NUMBERS

Let

$$\left(1 - \sum_{r=1}^{t} a_rx^r\right)^{-1} = 1 + \sum_{n=1}^{\infty} c(n,t)x^n\,,$$

where the a_r are rational integers.

In a recent paper (see [3]), it was proved that it is always possible to express the $c(n,t)$ by an explicit formula when $t = 1, 2, 3, 4$, and 5.

Then, using the methods in [3] we find the following Tribonacci formula ($T_n = c(n,3)$):

$$(39)\qquad T_n = \frac{x_1(x_3^{n+2} - x_2^{n+2}) + x_2(x_1^{n+2} - x_3^{n+2}) + x_3(x_2^{n+2} - x_1^{n+2})}{x_1(x_3^2 - x_2^2) + x_2(x_1^2 - x_3^2) + x_3(x_2^3 - x_1^2)} ,$$

where

$$x_1 = z_1 + 4/9z_1 + 1/3 ,$$
$$x_1 = z_2 + 4/9z_2 + 1/3 ,$$
$$x_3 = z_3 + 4/9z_3 + 1/3 ,$$

with

$$z_1 = (1/3)(3\sqrt{33} + 19)^{1/3} ,$$
$$z_2 = -(z_1/2)(1 - i\sqrt{3}) , \qquad (i = \sqrt{(-1)})$$
$$z_3 = -(z_1/2)(1 + i\sqrt{3}) ,$$

and

$$n = 0, 1, 2, \cdots .$$

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