FIBONACCI REPRESENTATIONS II

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1. Let R(N) denote the number of representations of

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_t}, \tag{1.1}$$

where

$$k_1 > k_2 > \cdots > k_t \geq 2. \tag{1.2}$$

The integer t is allowed to vary. We call (1.1) a Fibonacci representation of N provided (1.2) is satisfied. If in (1.1), we have

$$k_j - k_{j+1} \geq 2 \quad (j = 1, \cdots, t-1); \qquad k_t \geq 2, \tag{1.3}$$

then the representation (1.1) is unique and is called the canonical representation of N.

In a previous paper [1], the writer discussed the function R(N). The paper makes considerable use of the canonical representation and a function e(N) defined by

$$e(N) = F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_t-1}. \tag{1.4}$$

It is shown that e(N) is independent of the particular representation. The first main result of [1] is a reduction formula which theoretically enables one to evaluate R(N) for arbitrary N. Unfortunately, the general case is very complicated. However, if all the k_i in the canonical representation have the same parity, the situation is much more favorable and much simpler results are obtained.

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In the present paper, we consider the function $R(t,N)$ which is defined as the number of representations (1.1) subject to (1.2) where now t is fixed. Again we find a reduction formula which theoretically enables one to evaluate $R(t,N)$ but again leads to very complicated results. However, if all the k_1 in the canonical representation have the same parity, the results simplify considerably. In particular, if

$$N = F_{2k_1} + \cdots + F_{2k_r} \qquad (k_1 > k_2 > \cdots > k_r \geq 1) ,$$

$$j_s = k_s - k_{s+1} \quad (1 \leq s < r); \quad j_r = k_r ,$$

$$f_r(t) = f(t; j_1, \cdots, j_r) = R(t,N) ,$$

$$F_r(x) = F(x; j_1, \cdots, j_r) = \sum_{t=1}^{\infty} f(t; j_1, \cdots, j_r)\, x^t ,$$

$$G_r(x) = F(x; j_1, \cdots, j_{r-1}, j_r + 1) ,$$

then we have

$$(1.5) \quad G_r(x) - \frac{x(1 - x^{j_r+1})}{1 - x} G_{r+1}(x) - x^{j_{r-1}+2} G_{r-2}(x) = 0 \qquad (r \geq 2) ,$$

where

$$G_0(x) = 1, \quad G_1(x) = \frac{x(1 - x^{j_1+1})}{1 - x} .$$

In particular, if $j_1 = \cdots = j_r$, then

$$\sum_{r=0}^{\infty} G_r(x) z^r = \left\{1 - [j + 1]\, xz + x^{j+2} z^2\right\}^{-1} ,$$

from which an explicit formula for $G_r(x)$ is easily obtained. Also the case

$$j_1 = \cdots = j_{r-1} = j, \qquad j_r = k$$

leads to simple results.

In the final section of the paper some further problems are stated.

2. Put

$$\Phi(a, x, y) = \prod_{n=1}^{\infty} (1 + ax^{F_n} y^{F_{n+1}}) . \tag{2.1}$$

Then

$$\Phi(a, x, xy) = \prod_{n=1}^{\infty} (1 + ay^{F_{n+1}} x^{F_{n+2}}) = \prod_{n=2}^{\infty} (1 + ay^{F_n} x^{F_{n+1}}) ,$$

so that

$$(1 + axy)\Phi(a, x, xy) = \Phi(a, y, x) .$$

Now put

$$\Phi(a, x, y) = \sum_{k,m,n=0}^{\infty} A(k, m, n)\, a^k x^m y^n . \tag{2.2}$$

Comparison of coefficients gives

$$A(k, m, n) = A(k, n - m, m) + A(k - 1, n - m, m - 1) , \tag{2.3}$$

where it is understood that $A(k,m,n) = 0$ when any of the arguments is negative.

In the next place, it is evident from the definition of $e(N)$ and $R(k,N)$ that

$$(2.4)\qquad \prod_{n=1}^{\infty}\left(1 + ax^{F_n}y^{F_{n+1}}\right) = \sum_{N=0}^{\infty} R(k, N)\, a^k x^{e(N)} y^N .$$

Comparing (2.4) with (2.1) and (2.2), we get

$$(2.5)\qquad R(k, N) = A(k, e(N), N) .$$

In particular, for fixed k, n,

$$(2.6)\qquad A(k, m, n) = 0 \qquad (m \neq e(n)) .$$

It should be observed that $A(k,e(n),n)$ may vanish for certain values of k and n. However, since

$$R(n) = \sum_{k=0}^{\infty} R(k, n) = \sum_{k=0}^{\infty} A(k, e(n), n) ,$$

it follows that, for fixed n, there is at least one value of k such that

$$A(k, e(n), n) \neq 0 .$$

If we take $m = e(n)$ in (2.3), we get

$$(2.7)\qquad R(t,N) = A(t, N - e(N), e(N)) + A(t - 1, N - e(N), e(N) - 1) .$$

Now let N have the canonical representation

$$(2.8)\qquad N = F_{k_1} + \cdots + F_{k_r} ,$$

with k_r odd. Then

$$\begin{aligned} e(N) &= F_{k_1-1} + \cdots + F_{k_r-1} , \\ N - e(N) &= F_{k_1-2} + \cdots + F_{k_r-2} . \end{aligned}$$

Since $k_r \geq 3$, it follows that

$$N - e(N) = e(e(N)) . \tag{2.9}$$

On the other hand, exactly as in [1], we find that

$$e(e(N) - 1) = N - e(N) - 1 .$$

It follows that

$$A(t, N - e(N), e(N) - 1) = 0 ,$$

and (2.7) reduces to

$$R(t, N) = A(t, e(e(N)) .$$

We have, therefore,

$$R(t, N) = R(t, e(N)) \qquad (k_r \text{ odd}) . \tag{2.10}$$

Now let k_r in the canonical representation of N be even. We shall show that

$$R(t, N) = R(t - 1, e^{k_r - 1}(N_1)) + \sum_{j=2}^{s} R(t - j, e^{k_r - 2}(N_1)) , \tag{2.11}$$

where $k_r = 2s$,

$$N_1 = F_{k_1} + \cdots + F_{k_{r-1}} , \tag{2.12}$$

and

$$e^k(N) = e(e^{k-1}(N)), \qquad e^0(N) = N . \tag{2.13}$$

Assume first that $s > 1$. Then as above

$$N - e(N) = e(e(N)), \tag{2.14}$$

and

$$e(e(N) - 1) = e(e(N)). \tag{2.15}$$

Thus (2.7) becomes

$$R(t, N) = R(t, e(N)) + R(t - 1, e(N) - 1) \qquad (k_r > 2). \tag{2.16}$$

When $k_r = 2$, we have, as in [1],

$$N - e(N) = F_{k_1-2} + \cdots + F_{k_{r-1}-2} = e(e(N_1)),$$
$$e(N) - 1 = F_{k_1-1} + \cdots + F_{k_{r-1}-1} = e(N_1),$$
$$e(e(N)) = N - e(N) - 1.$$

It follows that

$$R(t, N) = R(t - 1, e(N_1)) \qquad (k_r = 2). \tag{2.17}$$

Returning to (2.16), since

$$e(N) - 1 = F_{k_1-1} + \cdots + F_{k_{k-1}-1} + (F_2 + F_4 + \cdots + F_{2t-2})$$
$$= e(N_1) + (F_2 + F_4 + \cdots + F_{2t-2}),$$

it follows from (2.17) and (2.10) that

$$R(t, e(N) - 1) = R(t - 1, e^2(N_1) + F_3 + \cdots + F_{2t-3})$$
$$= R(t - 1, e^3(N_1) + F_2 + \cdots + F_{2t-4}).$$

Repeating this process, we get

$$R(t,\ e(N) - 1) = R(t - s,\ e^{2s-2}(N_1))\ ,$$

so that (2.16) becomes

$$R(t, N) = R(t, e^2(N)) + R(t - s,\ e^{2s-2}(N_1)) \qquad (k_r = 2s > 2)\ . \tag{2.18}$$

If $k_r = 4$, Eq. (2.18) reduces, by (2.17) and (2.10), to

$$R(t,\ N) = R(t - 1,\ e^4(N_1)) + R(t - 2,\ e^2(N_1))\ ,$$

since

$$R(t,\ N) = R(t,\ e(N_1)) \qquad (k_r = 2)\ . \tag{2.19}$$

For $k_4 = 2s > 4$, Eq. (2.18) gives

$$\begin{aligned} R(t,\ N) &= R(t,\ e^4(N) + R(t - s + 1,\ e^{2s-2}(N_1)) + R(t - s,\ e^{2s-2}(N_1)) \\ &= R(t,\ e^6(N)) + R(t - s + 2,\ e^{2s-2}(N_1)) + R(t - s + 1,\ e^{2s-2}(N_1)) \\ &\qquad + R(t - s,\ e^{2s-2}(N_1))\ . \end{aligned}$$

Continuing in this way, we ultimately get

$$R(t,\ N) = R(t,\ e^{2s-2}(N)) + \sum_{j=2}^{s} R(t - j,\ e^{2s-2}(N_1))\ . \tag{2.20}$$

By (2.17),

$$R(t,\ e^{2s-2}(N)) = R(t - 1,\ e^{2s-1}(N_1))\ ,$$

so that (2.20) reduces to (2.11).

This proves (2.11) when $k_r > 2$; for $k_r = 2$, it is evident that (2.11) is identical with (2.17).

We may now state

Theorem 1. Let N have the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r} ,$$

where

$$k_j - k_{j+1} \geq 2 \quad (j = 1, \cdots, r - 1); \quad k_r \geq 2 .$$

Then, for $r > 1$, $t > 1$,

$$(2.21) \qquad R(t, N) = R(t - 1, e^{k_r - 1}(N_1)) + \sum_{j=2}^{s} R(t - j, e^{k_r - 2}(N_1)) ,$$

where $s = [k_r/2]$, $N_1 = F_{k_1} + \cdots + F_{k_{r-1}}$.

3. For $N = F_r$, $r \geq 2$, Eq. (2.7) reduces to

$$(3.1) \qquad \begin{aligned} R(t, F_r) &= A(t, F_{r-2}, F_{r-1}) + A(t - 1, F_{r-2}, F_{r-1} - 1) \\ &= R(t, F_{r-1}) + A(t - 1, F_{r-2}, F_{r-1} - 1) . \end{aligned}$$

Also,

$$(3.2) \qquad \begin{aligned} R(t, F_r - 1) &= A(t, F_r - 1 - e(F_r - 1), e(F_r - 1)) \\ &\quad + A(t - 1), F_r - 1 - e(F_r - 1), e(F_r - 1) - 1) . \end{aligned}$$

Since

$$e(F_{2s+1} - 1) = F_{2s}, \quad e(F_{2s} - 1) = F_{2s-1} - 1 ,$$

we have

$$\begin{aligned} A(t - 1, F_{2s-2}, F_{2s-1} - 1) &= R(t - 1, F_{2s-1} - 1), \\ A(t - 1, F_{2s} - 1) &= 0 . \end{aligned}$$

Thus (3.1) becomes

$$\text{(3.2)}\qquad \begin{cases} R(t, F_{2s}) = R(t, F_{2s-1}) + R(t-1, F_{2s-1} - 1), \\ R(t, F_{2s-1}) = R(t, F_{2s}) . \end{cases}$$

In the next place, Eq. (3.2) gives

$$\begin{aligned} R(t, F_{2s} - 1) &= A(t, F_{2s-2}, F_{2s-1} - 1) + A(t-1, F_{2s-2}, F_{2s-1} - 2) \\ &= R(t, F_{2s-1} - 1), \\ R(t, F_{2s+1} - 1) &= A(t, F_{2s-1} - 1, F_{2s}) + A(t-1, F_{2s-1} - 1, F_{2s} - 1) \\ &= R(t-1, F_{2s} - 1), \end{aligned}$$

that is,

$$\text{(3.3)}\qquad R(t, F_r - 1) = R(t - \lambda, F_{r-1} - 1) \qquad (r \geq 2),$$

where

$$\lambda = \begin{cases} 0 & (r \text{ even}) \\ 1 & (r \text{ odd}) . \end{cases}$$

It follows from (3.3) that

$$R(t, F_{2s} - 1) = R(t - s + 1, 0), \quad R(t, F_{2s+1} - 1) = R(t - s + 1, 1)$$

which gives

$$\text{(3.4)}\qquad \begin{cases} R(t, F_{2s} - 1) = \delta_{t,s-1} \\ R(t, F_{2s+1} - 1) = \delta_{t,s} . \end{cases}$$

Combining (3.2) with (3.4), we get

$$R(t, F_{2s}) = R(t, F_{2s+1}) = R(t, F_{2s-1}) + \delta_{t,s},$$

so that

$$R(t, F_{2s}) = R(t, F_{2s-2}) + \delta_{t,s} .$$

It follows that

$$R(t, F_{2s}) = \begin{cases} 1 & (1 \le t \le s) \\ 0 & (t > s) \end{cases} .$$

We may now state

Theorem 2. We have, for $s \geq 1$, $t \geq 1$,

$$\begin{aligned} &R(t, F_{2s+1} - 1) = R(t, F_{2s+2} - 1) = \delta_{t,s} , \\ (3.5) \qquad &R(t, F_{2s}) = R(t, F_{2s+1}) = \begin{cases} 1 & (1 \le t \le s) \\ 0 & (t > s) \end{cases} . \end{aligned}$$

Let $m(N)$ denote the minimum number of summands in a Fibonacci representation of N and let $M(N)$ denote the maximum number of summands. It follows at once from (2.21) that

$$(3.6) \qquad m(N) = r ,$$

where r is the number of summands in the canonical representation of N. Moreover, it is easily proved by induction that

$$(3.7) \qquad R(r, N) = 1.$$

As for $M(N)$, it follows from (2.21) that

$$(3.8) \qquad M(N) \le M(F_{k_1-k_2+2} + \cdots + F_{k_{r-1}-k_r+2}) + [\tfrac{1}{2}k_r] ,$$

where

$$N = F_{k_1} + \cdots + F_{k_r}$$

is the canonical representation. Now, by Theorem 2,

$$M(F_k) = [\tfrac{1}{2}k] .$$

Hence by (3.8),

$$M(F_{k_1} + F_{k_2}) \le [\tfrac{1}{2}(k_1 - k_2)] + [\tfrac{1}{2}k_2] + 1 .$$

Again, applying (3.8), we get

$$M(F_{k_1} + F_{k_2} + F_{k_3}) \le [\tfrac{1}{2}(k_1 - k_2)] + [\tfrac{1}{2}(k_2 - k_3)] + [\tfrac{1}{2}k_2] + 2 .$$

It is clear that in general we have

$$M(N) \le [\tfrac{1}{2}(k_1 - k_2)] + \cdots + [\tfrac{1}{2}(k_{r-1} - k_r)] + [\tfrac{1}{2}k_r] + r - 1 , \tag{3.9}$$

so that

$$M(N) \le [\tfrac{1}{2}k_1] + r - 1 . \tag{3.10}$$

We note also that (2.21) implies

$$R(M(N), N) = 1 . \tag{3.11}$$

We may state

Theorem 3. Let

$$N = F_{k_1} + \cdots + F_{k_r} \tag{3.12}$$

be the canonical representation of N. Let $m(N)$ denote the minimum number of summands in any Fibonacci representation of N and let $M(N)$ denote the maximum number of summands. Then $m(N) = r$ and $M(N)$ satisfies (3.9). Moreover,

$$R(m(N), N) = R(M(N), N) = 1 . \tag{3.13}$$

It can be shown by examples that (3.9) need not be an equality when $r > 1$.

4. While Theorem 1 theoretically enables one to compute $R(t,N)$ for arbitrary t,N, the results are usually very complicated. Simpler results can be obtained when the k_j in the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r} \tag{4.1}$$

have the same parity. In the first place, if all the k_j are odd, then, by (2.10),

$$R(t, F_{k_1} + \cdots + F_{k_r}) = R(t, F_{k_1-1} + \cdots + F_{k_r-1}) .$$

There is therefore no loss in generality in assuming that all the k_j are even

It will be convenient to use the following notation. Let N have the canonical representation

$$N = F_{2k_1} + \cdots + F_{2k_r} , \tag{4.2}$$

where

$$k_1 > k_2 > \cdots > k_r \geq 1 . \tag{4.3}$$

Then, by (2.21) and (2.10),

$$\begin{aligned} R(t,N) = {} & R(t - 1, F_{2k_1-2k_r} + \cdots + F_{2k_{r-1}-2k_r}) \\ & + \sum_{j=2}^{k_r} R(t - j, F_{2k_1-2k_r+2} + \cdots + F_{2k_{r-1}-2k_r+2}). \end{aligned} \tag{4.4}$$

Put

$$j_s = k_s - k_{s-1} \quad (s = 1, \cdots, r - 1); \quad j_r = k_r \tag{4.5}$$

and

$$f_r(t) = f(t; j_1, \cdots, j_r) = R(t, N) . \tag{4.6}$$

Then (4.4) becomes

$$(4.7)\qquad f(t;\, j_1, \cdots, j_r) = f(t-1;\, j_1, \cdots, j_{r-1}) + \sum_{u=2}^{j_r} f(t-u;\, j_1, \cdots, j_{r-2}, j_{r-1}+1)\,.$$

By (2.18), we have

$$\begin{aligned} R(t,\, F_{2k_1-2k_r+2} + \cdots + F_{2k_{r-1}-2k_r+2}) &= R(t,\, F_{2k_1-2k_r} + \cdots + F_{2k_{r-1}-2k_r}) \\ &\quad + R(t-k_{r-1}+k_r-1;\, F_{2k_1-2k_{r-1}+2} + \cdots \\ &\quad + F_{2k_{r-2}-2k_{r-1}+2})\,, \end{aligned}$$

so that

$$\begin{aligned} (4.8)\qquad & f(t;\, j_1, \cdots, j_{r-2}, j_{r-1}+1) \\ &= f(t;\, j_1, \cdots, j_{r-2}, j_{r-1}) + f(t-j_{r-1}-1;\, j_1, \cdots, j_{r-3}, j_{r-2}+1). \end{aligned}$$

If we put

$$(4.9)\qquad F_r(x) = F(x;\, j_1, \cdots, j_r) = \sum_{t=1}^{\infty} f(t;\, j_1, \cdots, j_r)\, x^t\,,$$

it follows from (4.7) that (for $r > 1$),

$$(4.10)\qquad F(x;\, j_1, \cdots, j_r) = xF(x;\, j_1, \cdots, j_{r-1}) + \frac{x(x - x^{j_r})}{1-x} F(x;\, j_1, \cdots, j_{r-2}, j_{r-1}+1).$$

Similarly, by (4.8),

$$(4.11)\qquad F(x;\, j_1, \cdots, j_{r-2}, j_{r-1}+1) = F(x;\, j_1, \cdots, j_{r-2}, j_{r-1}) + x^{j_{r-1}+1} F(x;\, j_1, \cdots, j_{r-3}, j_{r-2}+1),$$

which yields

$$(4.12)\qquad F(x;\, j_1, \cdots, j_{r-2}, j_{r-1}+1) = F(x;\, j_1, \cdots, j_{r-2}, j_{r-1}) + x^{j_{r-1}+1} F(x;\, j_1, \cdots, j_{r-3}, j_{r-2}) + x^{j_{r-1}+j_{r-2}+2} F(x;\, j_1, \cdots, j_{r-3}) + \cdots + x^{j_{r-1}+\cdots+j_2+r-1} F(x;\, j_1).$$

For brevity, put

$$(4.13)\qquad G_r(x) = F(x;\, j_1, \cdots, j_{r-1}, j_r+1),$$

so that (4.10) becomes

$$(4.14)\qquad F_r(x) - xF_{r-1}(x) = \frac{x(x - x^{j_r})}{1 - x} G_{r-1}(x),$$

while (4.11) becomes

$$(4.15)\qquad G_{r-1}(x) = F_{r-1}(x) + x^{j_{r-1}+1} G_{r-2}(x).$$

Combining (4.14) with (4.15), we get

$$(4.16)\qquad G_r(x) - \frac{x(1 - x^{j_{r+1}})}{1 - x} G_{r-1}(x) + x^{j_{r-1}+2} G_{r-2}(x) = 0.$$

Thus $G_r(x)$ satisfies a recurrence of the second order. Note that

$$G_1(x) = F(x;\, j_1 + 1) = \sum_{t=1}^{\infty} R(t, F_{2j_1+2})\, x^t = \sum_{t=1}^{j_1+1} x^t = \frac{x(1 - x^{j_1+1})}{1 - x},$$

$$G_2(x) = F(x; j_1, j_2 + 1) = \sum_{t=2}^{\infty} R(t, F_{2j_1+2j_2+2} + F_{2j_2+2}) .$$

Now, by (2.21),

$$R(t, F_{2j_1+2j_2+2} + F_{2j_2+2}) = R(t - 1, F_{2j_1+1}) + \sum_{u=2}^{j_2+1} R(t - u, F_{2j_1+2}) ,$$

so that

$$G_2(x) = x \sum_{t=1}^{j_1} x^t + \sum_{u=2}^{j_2+1} x^u \sum_{t=1}^{j_1+1} x^t$$

$$= \frac{x^2(1 - x^{j_1})}{1 - x} + \frac{x^2(1 - x^{j_2})}{1 - x} \frac{x(1 - x^{j_1+1})}{1 - x} .$$

Hence, if we take $G_0(x) = 1$, Eq. (4.16) holds for all $r \geq 2$.

We may state

Theorem 5. With the notation (4.2), (4.6), (4.9), (4.12), $f_r(t) = R(t,N)$ is determined by means of the recurrence (4.16) with

$$G_0(x) = 1, \qquad G_1(x) = \frac{x(1 - x^{j_1+1})}{1 - x}$$

and

$$F_r(x) = G_r(x) - x^{j_r+1} G_{r-1}(x) .$$

It is easy to show that $G_r(x)$ is equal to the determinant

$$(4.17)\qquad D_r(x) = \begin{vmatrix} x[j_1+1] & -x^{j_1+2} & 0 & \cdots & 0 \\ -1 & x[j_2+1] & -x^{j_2+2} & \cdots & 0 \\ 0 & & x[j_3+1] & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & x[j_r+1] \end{vmatrix},$$

where

$$(4.18)\qquad [j] = (1 - x^j)/(1 - x) .$$

Indeed,

$$D_1(x) = x[j_1 + 1] = G_1(x) ,$$

$$D_2(x) = x^2[j_1 + 1][j_2 + 1] - x^{j_1+2} = x^3[j_1 + 1][j_2] + x^2[j_1] = G_2(x) ,$$

and

$$(4.19)\qquad D_r(x) = x[j_r + 1]D_{r-1}(x) - x^{j_{r-1}+2} D_{r-2}(x) .$$

Since the recurrence (4.16) and (4.19) are the same, it follows that $G_r(x) = D_r(x)$.

5. When

$$(5.1)\qquad j_1 = j_2 = \cdots = j_r = j ,$$

we can obtain an explicit formula for $G_r(x)$. The recurrence (4.16) reduces to

$$(5.2)\qquad G_r(x) - x[j - 1]G_{r-1}(x) + x^{j+2}G_{r-2}(x) = 0 \qquad (r \geq 2) .$$

Then

$$\sum_{r=0}^{\infty} G_r(x)z^r = 1 + [j+1]xz + \sum_{r=2}^{\infty} G_r(x)z^r$$

$$= 1 + [j+1]xz + \sum_{r=2}^{\infty} \left\{x[j+1]G_{r-1}(x) - x^{j+2}G_{r-2}(x)\right\} z^r$$

$$= 1 + ([j+1]xz + x^{j+2} z^2) \sum_{r=0}^{\infty} G_r(x) z^r ,$$

so that

$$\sum_{r=0}^{\infty} G_r(x) z^r = (1 - [j+1] xz + x^{j+2} z^2)^{-1}$$

$$= \sum_{s=0}^{\infty} x^s z^s ([j+1] - x^{j+1} z)^s$$

$$= \sum_{s=0}^{\infty} x^s z^s \sum_{t=0}^{s} (-1)^t \binom{s}{t} [j+1]^{s-t} x^{(j+1)t} z^t .$$

Hence

$$G_r(x) = \sum_{2t \leq r}^{\infty} (-1)^t \binom{r-t}{t} [j+1]^{r-2t} x^{r+jt} . \tag{5.3}$$

Finally, we compute $F_r(x)$ by using

$$F_r(x) = G_r(x) - x^{j+1} G_{r-1}(x) . \tag{5.4}$$

When $j = 1$, we have

$$\sum_{r=0}^{\infty} G_r(x)^r = \frac{1}{(1 - xz)(1 - x^2z)} = \frac{1}{1 - x}\left(\frac{1}{1 - xz} - \frac{1}{1 - x^2z}\right) ,$$

which gives

$$(5.5) \qquad G_r(x) = x^2[r] = \frac{x^r(1 - x^r)}{1 - x} \qquad (j = 1;\ r \geq 1)$$

$$(5.6) \qquad F_r(x) = x^r \qquad (j = 1) .$$

In this case, we evidently have

$$N = F_{2r} + F_{2r-2} + \cdots + F_2 = F_{2r+1} - 1 ,$$

so that (5.6) is in agreement with (3.4).

For certain applications, it is of interest to take

$$(5.7) \qquad j_1 = \cdots = j_{r-1} = j; \qquad j_r = k .$$

Then $G_1(x)$, $G_2(x)$, $\cdots$, $G_{r-1}(x)$ are determined by

$$(5.8) \qquad G_s(x) = \sum_{2t \leq s} (-1)^t \binom{s - t}{t} [j - 1]^{s-2t} x^{s+jt} \qquad (1 \leq s < r),$$

while

$$(5.9) \qquad G'_r(x) = x[k - 1]G_{r-1}(x) - x^{j+2}G_{r-2}(x) ,$$

where

$$G'_r(x) = G_r(x;\ j, \cdots, j, k) .$$

Also,

(5.10) $$F'_r(x) = F_r(x;\ j, \cdots, j, k) = x[k]G_{r-1}(x) - x^{j+2}G_{r-2}(x) .$$

We shall now make some applications of these results. Since

$$L_{2j+1}F_{2k} = F_{2k+2j} + F_{2k+2j-2} + \cdots + F_{2k-2j} ,$$

it follows from (5.10) that

(5.11) $$\sum_t R(t, L_{2j+1}F_{2k})x^t = x^{2j+1}[2j][k - j] - x^{2j+2}[2j - 1] \qquad (j < k) .$$

(Note that formula (6.17) of [1] should read

$$R(L_{2j+1}F_{2k}) = 2j(k - j) - (2j - 1)$$

in agreement with (5.11).) If we rewrite (5.11) as

$$\sum_t R(t, L_{2j+1}F_{2k})x^t = x^{2j+1}\{1 + x + \cdots + x^{k-j-1} + (x + \cdots + x^{2j-1})(x + \cdots + x^{k-j-1})\}$$

we can easily evaluate $R(t, L_{2j+1}F_{2k})$. In particular, we note that

(5.12) $$R(t, L_{2j+1}F_{2k}) > 0 \qquad (j < k)$$

if and only if

$$2j + 1 \le t \le 3j + k - 1 .$$

Note that, for $k = 3j$,

$$\sum_t R(t, L_{2j+1}F_{6j})x^t = x^{2j+1}\{1 + x + \cdots + x^{2j-1} + (x + x^2 + \cdots + x^{2j-1})^2\} .$$

This example shows that the function $R(t, N)$ takes on arbitrarily large values.

When $j = k$, we have

$$L_{2k+1}F_{2k} = F_{4k+1} - 1 ,$$

so that, by (3.4),

$$\sum_t R(t, L_{2k+1}F_{2k})\, x^t = x^{2k} . \tag{5.13}$$

Next, since

$$L_{2j+1}F_{2k} = F_{2j+2k} + F_{2j+2k-2} + \cdots + F_{2j-2k-2} \qquad (j > k) ,$$

we get

$$\sum_t R(t, L_{2j+1}F_{2k})x^t = x^{2k}[j - k - 1]\,[2k - 1] - x^{2k+1}[2k - 2] \qquad (j > k > 1) . \tag{5.14}$$

Corresponding to (5.15), we now have

$$R(t, L_{2j+1}F_{2k}) > 0 \qquad (j > k > 1) , \tag{5.15}$$

if and only if

$$2k \le t \le j + 3k - 2 .$$

The case $k = 1$ is not included in (5.14), because (5.5) does not hold when $r = 0$. For the excluded case, since

$$L_{2j+1} = F_{2j+2} + F_{2j} ,$$

we get, by Theorem 1,

(5.16) $$\sum_t R(t,\ L_{2j+1})x^t = x^2 + (x^2 + x^3)\frac{x - x^j}{1 - x} \qquad (j \geq 1)\ .$$

For $t = 1$, Eq. (5.16) reduces to the known result:

$$R(L_{2j+1}) = 2j - 1\ .$$

In [1] a number of formulas of the type

$$R(F^2_{2n+1} - 1) = F_{2n+1} \quad (n \geq 0), \qquad R(F^2_{2n}) = F_{2n} \quad (n \geq 1)$$

were obtained. They depend on the identities

$$F_4 + F_8 + \cdots + F_{4n} = F^2_{4n+1} - 1\ ,$$
$$F_2 + F_6 + \cdots + F_{4n+2} = F^2_{2n}\ .$$

We now apply (5.10) to these identities. Then $G_r(x)$ is determined by

(5.17) $$G_r(x) = \sum_{2t \leq r} (-1)^t \binom{r - t}{t} [3]^{r-2t} x^{r+2t}\ .$$

Thus (5.10) yields

(5.18) $$\sum_t R(t,\ F^2_{4n+1} - 1)x^t = x(1 + x)G_{n-1}(x) - x^4 G_{n-2}(x)\ ,$$

(5.19) $$\sum_t R(t,\ F^2_{2n})x^t = xG_{n-1}(x) - x^4 G_{n-2}(x)\ ,$$

with $G_{n-1}(x)$, $G_{n-2}(x)$ given by (5.17) .

It may be of interest to note that

$$G_r(1) = \sum_{2t \le r} (-1)^t \binom{r-t}{t} 3^{r-2t} = F_{2r+2} .$$

6. The following problems may be of some interest.

   A. Evaluate $M(N)$ in terms of the canonical representation of N.

   B. Determine whether $R(t,N) \ge 1$ for all t in $m(N) \le t \le M(N)$.

   C. Does $R(t,N)$ have the unimodal property? That is, for given N, does there exist an integer $\mu(N)$ such that

$$R(t, N) \le R(t+1, N) \qquad (m(N) \le t \le \mu(N)),$$
$$R(t, N) \ge R(t+1, N) \qquad (\mu(N) \le t < M(N)) ?$$

   D. Is $R(t, N)$ logarithmically concave? That is, does it satisfy

$$R^2(t,N) \ge R(t-1,N)R(t+1,N) \qquad (m(N) < t < M(N))?$$

   E. Find the general solution of the equation

$$R(t, N) = 1 .$$

REFERENCES

1. L. Carlitz, "Fibonacci Representations," Fibonacci Quarterly, Vol. 6 (1968), pp. 193-220.
2. D. A. Klarner, "Partitions of N into Distinct Fibonacci Numbers," Fibonacci Quarterly, Vol. 6 (1968), pp. 235-244.