SOME PROPERTIES OF STIRLING NUMBERS OF THE SECOND KIND

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INTRODUCTION

In attempting to predict the number of demands that will occur during a given period of time, for supplies in military inventory systems, it becomes necessary to formulate suitable probability models for the distribution of demands of individual items of supply. One such model, described in [1], involves two parameters, to be estimated from available data. For example, in the case of predicting demands for items installed on Polaris submarines, the data might consist of items demanded in a series of patrols.

In studying the properties of estimation procedures for parameters of any model, one is led to a consideration of the sampling distributions of the estimates. For the model described in [1], the sampling distributions of some proposed estimates were found to involve Stirling numbers of the second kind, and in the derivation of these distributions from the initial probability assumptions, some properties of these numbers become of interest.

A Stirling number of the second kind, $\mathcal{S}_T^{(m)}$, is the number of ways of partitioning a set of T elements into m non-empty subsets. Thus, if we have the set of elements (1, 2, 3) with T = 3 and m = 2, we have
sible partitions

$$(1,2),\quad (3)$$
$$(1,3),\quad (2)$$
$$(2,3),\quad (1)$$

with

$$\mathcal{S}_3(2) = 3 \ .$$

If the order of the partitions is taken into account, that is,

$$(1,2),\quad (3)\ ,$$

and

$$(3), \quad (1,2)$$

are considered to be two partitions, the number of ordered partitions is

$$m!\mathcal{S}_T^{(m)} \ .$$

For example, suppose that a given item installed in a Polaris submarine is demanded in each of m partols, with a total quantity demanded of T units, $(T \geq m)$. The number of different ways of partitioning T into m demands is $\mathcal{S}_T^{(m)}$; the number of ways in which a particular partition can be assigned to the m patrols is m! ; thus the number of possible assignments of the total quantity demanded to the m patrols is $m!\mathcal{S}_T^{(m)}$.

PROPERTIES OF STIRLING NUMBERS OF THE SECOND KIND

The generating function of Stirling numbers of the second kind is

$$x^T = \sum_{m=0}^{T} \mathcal{S}_T^{(m)} x(x-1)\cdots(x-m+1) \ .$$

In closed form,

$$\mathcal{S}_T^{(m)} = \frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}k^T \ .$$

Various properties of these numbers are known (e.g., see [2]). Thus,

$$\begin{aligned} \mathcal{S}_T^{(m)} &= 0 \quad \text{for} \quad T < m \\ \mathcal{S}_T^{(T)} &= 1 \quad \text{for} \quad T = 0, 1, 2, \cdots \\ \mathcal{S}_T^{(0)} &= 0 \quad \text{for} \quad T = 1, 2, \cdots \\ \mathcal{S}_T^{(1)} &= 1 \quad \text{for} \quad T = 1, 2, \cdots \ . \end{aligned}$$

We have the recurrence relations

(2) $$\mathcal{S}_{T+1}^{(m)} = m\mathcal{S}_T^{(m)} + \mathcal{S}_T^{(m-1)} \quad \text{for} \quad T \geq m \geq 1$$

(3) $$\binom{m}{r}\mathcal{S}_T^{(m)} = \sum_{k=m-r}^{T-r} \binom{T}{k}\mathcal{S}_{T-k}^{(r)}\mathcal{S}_k^{(m-r)} \quad \text{for} \quad T \geq m \geq r\,.$$

If $r = 1$, Eq. (3) becomes

(4) $$m\mathcal{S}_T^{(m)} = \sum_{k=m-1}^{T-1} \binom{T}{k}\mathcal{S}_k^{(m-1)}$$

The following results appear to be less well known.

Lemma 1. For any integers r and k, with $k = 0, 1, \cdots$, and $r = k+1, k+2, \cdots$,

(5) $$\sum_{j=0}^{r} \binom{r+k}{j}(-1)^j \mathcal{S}_{r-j+k}^{(r-j)} = 0\,.$$

Proof. We prove the lemma by induction on k. In the proof, we use the recurrences

$$\binom{r+k}{j} = \binom{r+k-1}{j} + \binom{r+k-1}{j-1}$$

and

$$\mathcal{S}_{r-j+k}^{(r-j)} = (r-j)\mathcal{S}_{r-j+k-1}^{(r-j)} + \mathcal{S}_{r-j+k-1}^{(r-j-1)}$$

For k = 0 and r = 1, 2, ⋯,

$$\sum_{j=0}^{r} \binom{r}{j} (-1)^j \mathcal{S}_{r-j}^{(r-j)} = \sum_{j=0}^{r} \binom{r}{j} (-1)^j = (1 - 1)^r = 0 .$$

For k = 1 and r = 2, 3, ⋯,

$$\sum_{j=0}^{r} \binom{r+1}{j} (-1)^j \mathcal{S}_{r-j+1}^{(r-j)} = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \mathcal{S}_{r-j+1}^{(r-j)}$$

$$+ \sum_{j=1}^{r} \binom{r}{j-1} (-1)^j \mathcal{S}_{r-j+1}^{(r-j)}$$

$$= \sum_{j=0}^{r} \binom{r}{j} (-1)^j (r - j) \mathcal{S}_{r-j}^{(r-j)}$$

$$+ \sum_{j=0}^{r-1} \binom{r}{j} (-1)^j \mathcal{S}_{r-j}^{(r-j-1)} - \sum_{j=0}^{r-1} \binom{r}{j} (-1)^j \mathcal{S}_{r-j}^{(r-j-1)}$$

The last two terms cancel each other while the first one becomes

$$r \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j = 0 .$$

We suppose the result holds for $k = m - 1$, and $r = m, m + 1, \cdots$, and show that it holds for $k = m$, $r = m + 1, m + 2, \cdots$. We have

$$\sum_{j=0}^{r}\binom{r+m}{j}(-1)^j\mathcal{S}_{r-j+m}^{(r-j)}$$

$$= \sum_{j=0}^{r}\binom{r+m-1}{j}(-1)^j\mathcal{S}_{r-j+m}^{(r-j)}$$

$$+ \sum_{j=1}^{r}\binom{r+m-1}{j-1}(-1)^j\mathcal{S}_{r-j+m}^{(r-j)}$$

$$= \sum_{j=0}^{r}\binom{r+m-1}{j}(-1)^j(r-j)\mathcal{S}_{r-j+m-1}^{(r-j)}$$

$$+ \sum_{j=0}^{r-1}\binom{r+m-1}{j}(-1)^j\mathcal{S}_{r-j+m-1}^{(r-j-1)}$$

$$- \sum_{j=0}^{r-1}\binom{r+m-1}{j}(-1)^j\mathcal{S}_{r-j+m-1}^{(r-j-1)}$$

Again the last two terms cancel each other. The first term becomes

$$r\sum_{j=0}^{r}\binom{r+m-1}{j}(-1)^j\mathcal{S}_{r-j+m-1}^{(r-j)}$$

$$- (r+m-1)\sum_{j=0}^{r-1}\binom{(r-1)+m-1}{j}(-1)^j\mathcal{S}_{(r-1)-j+m-1}^{(r-1-j)}$$

Since $r \geq m+1$ and $(r-1) \geq m$, both of these sums are zero, giving the desired result.

<u>Lemma 2.</u> For any integers m and T such that $T \geq 2m$

(6)
$$\sum_{k=2(m-1)}^{T-2} \binom{T}{k} (m-1)! \sum_{j=0}^{m-1} \binom{k}{j} (-1)^j \mathcal{S}_{k-j}^{(m-1-j)} = m! \sum_{j=0}^{m} \binom{T}{j} (-1)^j \mathcal{S}_{T-j}^{(m-j)} .$$

Proof.

$$\sum_{k=2(m-1)}^{T-2} \binom{T}{k} (m-1)! \sum_{j=0}^{m-1} \binom{k}{j} (-1)^j \mathcal{S}_{k-j}^{(m-1-j)}$$

$$= (m-1)! \sum_{j=0}^{m-1} \binom{T}{j} (-1)^j \sum_{(k-j)=2(m-1)-j}^{T-j-2} \binom{T-j}{k-j} \mathcal{S}_{k-j}^{(m-j-1)}$$

From Eq. (4),

$$\sum_{(k-j)=m-j-1}^{T-j-1} \binom{T-j}{k-j} \mathcal{S}_{k-j}^{(m-j-1)} = (m-j) \mathcal{S}_{T-j}^{(m-j)} .$$

It follows that

$$\sum_{(k-j)=2(m-1)-j}^{T-j-2} \binom{T-j}{k-j} \mathcal{S}_{k-j}^{(m-j-1)} = \sum_{(k-j)=m-j-1}^{T-j-1} \binom{T-j}{k-j} \mathcal{S}_{k-j}^{(m-j-1)}$$

$$- \binom{T-j}{T-j-1} \mathcal{S}_{T-j-1}^{(m-j-1)}$$

$$- \sum_{(k-j)=m-j-1}^{2m-j-3} \binom{T-j}{k-j} \mathcal{S}_{k-j}^{(m-j-1)}$$

$$= (m-j) \mathcal{S}_{T-1}^{(m-j)} - (T-j) \mathcal{S}_{T-j-k}^{(m-j-1)}$$

$$- \sum_{r=m-j-1}^{2m-j-3} \binom{T-j}{r} \mathcal{S}_{r}^{(m-1-j)} .$$

The right-hand side of Eq. (7) becomes

$$m!\sum_{j=0}^{m}\binom{T}{j}(-1)^j\mathscr{S}_{T-j}^{(m-j)} - (m-1)!\left[\sum_{j=1}^{m-1}\frac{T!\,(-1)^j}{(j-1)!\,(T-j)!}\mathscr{S}_{T-j}^{(m-j)} + \sum_{j=0}^{m-2}\frac{T!\,(-1)^j}{j!\,(T-1-j)!}\mathscr{S}_{T-1-j}^{(m-1-j)}\right]$$

$$- (m-1)!\sum_{j=0}^{m-1}\binom{T}{j}(-1)^j\sum_{r=m-1-j}^{2m-3-j}\binom{T-j}{r}\mathscr{S}_r^{(m-1-j)}$$

The two sums in brackets cancel each other since

$$\sum_{j=1}^{m-1}\frac{T!\,(-1)^j}{(j-1)!\,(T-j)!}\mathscr{S}_{T-j}^{(m-j)} = -\sum_{j=0}^{m-2}\frac{T!\,(-1)^j}{j!\,(T-1-j)!}\mathscr{S}_{T-1-j}^{(m-1-j)} .$$

In the final term, we set $k = (m-1-j)$ so that k ranges from zero to $(m-2)$. Interchanging the order of summation and rewriting the expression, we have

$$(m-1)!\sum_{j=0}^{m-1}\binom{T}{j}(-1)^j\sum_{r=m-1-j}^{2m-3-j}\binom{T-j}{r}\mathscr{S}_r^{(m-1-j)}$$

$$= \sum_{k=0}^{m-2}\binom{T}{m-1+k}\left[\sum_{j=0}^{m-1}\binom{m-1+k}{j}(-1)^j\mathscr{S}_{m-1-j+k}^{(m-1-j)}\right]$$

From Lemma 1, each of the inner sums is zero, so that Eq. (8) becomes

$$m! \sum_{j=0}^{m} \binom{T}{j} (-1)^j \mathcal{S}_{T-j}^{(m-j)}$$

and the lemma follows.

These properties are useful in proving the following theorems.

Theorem 1. Let $t_1, t_2, \cdots, t_m$ be m integers such that

$$t_i \geq 1 \quad i = 1, 2, \cdots, m$$

and

$$\sum_{i=1}^{m} t_i = T \geq m \ .$$

Then

$$\sum_{t_1} \sum_{t_2} \cdots \sum_{t_m} \frac{T!}{\prod_{i=1}^{m} t_i!} = m! \mathcal{S}_T^{(m)} \tag{9}$$

Proof. We write

$$\begin{aligned} t_1 &= T_1 \\ t_1 + t_2 &= T_2 \\ &\vdots \\ t_1 + t_2 + \cdots + t_m &= T_m = T \ . \end{aligned}$$

The summation in (9) can be rewritten as

$$\sum_{T_{m-1}}^{T} = m - 1 \binom{T}{T_{m-1}} \left[\cdots \left[\sum_{T_2=2}^{T_3-1} \binom{T_3}{T_2} \left[\sum_{T_1=1}^{T_2-1} \binom{T_2}{T_1} \right] \right] \cdots \right] \ .$$

But

$$\sum_{T_1=1}^{T_2-1} \binom{T_2}{T_1} = 2^{T_2} - 2 = 2!\, \mathcal{S}_{T_2}^{(2)} \quad .$$

From Eq. (4),

$$2! \sum_{T_2=2}^{T_3-1} \binom{T_3}{T_2} \mathcal{S}_{T_2}^{(2)} = 3!\, \mathcal{S}_{T_3}^{(3)} ,$$

and, in general,

$$(r - 1)! \sum_{T_r=(r-1)}^{T_r-1} \binom{T_r}{T_{r-1}} \mathcal{S}_{T_{r-1}}^{(r-1)} = r!\, \mathcal{S}_{T_r}^{(r)} ,$$

and the theorem follows.

Theorem 2. Let $t_1, t_2, \cdots, t_m$ be m integers such that

$$t_i \geq 2 \quad i = 1, 2, \cdots, m$$

and

$$\sum_{i=1}^{m} t_i = T \geq 2m ,$$

then

$$\sum_{t_1}\sum_{t_2}\cdots\sum_{t_m} \frac{T!}{\prod_{i=1}^{m} t_i!} = m!\sum_{j=0}^{m}\binom{T}{j}(-1)^j \mathcal{S}_{T-j}^{(m-j)} \tag{10}$$

<u>Proof</u>. Again, let

$$\begin{aligned} t_2 &= T_1 \\ t_1 + t_2 &= T_2 \\ &\vdots \\ t_1 + t_2 + \cdots + t_m &= T_m = T\ . \end{aligned}$$

The desired summation can now be written as

$$\sum_{T_{m-1}=2(m-1)}^{T}\binom{T}{T_{m-1}}\left[\cdots\left[\sum_{T_2=4}^{T_3-2}\binom{T_3}{T_2}\left[\sum_{T_1=2}^{T_2-2}\binom{T_2}{T_1}\right]\right]\cdots\right]$$

$$\sum_{T_1=2}^{T_2-2}\binom{T_2}{T_1} = 2^{T_2} - 2 - 2^{T_2}$$

$$= 2!\sum_{j=0}^{2}\mathcal{S}_j^{T_2}\,(-1)^j\,\mathcal{S}_{T_2-j}^{(2-j)}\ .$$

Repeated application of Lemma 2 leads to the desired result.

REFERENCES

1. R. Sitgreaves, "Estimation Procedures for a Generalized Poisson," (in preparation).
2. Milton Abramowitz and Irene A. Stegun (ed.), <u>Handbook of Mathematical Functions</u>, U. S. Department of Commerce, National Bureau of Standards, Applied Mathematics (1964), 55, pp. 825-826.