SOME UNIVERSAL COUNTEREXAMPLES

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In [1], H. H. Ferns discussed minimal and maximal representations of positive integers as sums of distinct Fibonacci numbers. S. G. Mohanty extended those results in [2] by employing a one-parameter family of generalized Fibonacci sequences. This paper provides clarification of the concepts of maximality and minimality as employed by Ferns and Mohanty.

For convenience we will reiterate several definitions and results from [2], with suitably altered notation.

Definition 1: The generalized Fibonacci sequence $\{U_{r,n}\}$ with parameter r is given by

$$U_{r,1} = U_{r,2} = \cdots = U_{r,r} = 1$$
$$U_{r,n} = U_{r,n-1} + U_{r,n-r}$$

for integers n and r such that $n > r > 1$.

For brevity, the parameter r will not be made explicit. Thus $U_{r,n} = U_n$ and $\{U_{r,n}\} = \{U_n\}$. Since we wish to express positive integers as sums of numerically distinct terms of $\{U_n\}$, we make the restriction that the first $r - 1$ terms not be employed in any representation. After Mohanty, we assert without proof that every positive integer N has at least one representation in $\{U_n\}$ subject to that restriction. That is, there exist integers a_i such that $a_i = 0$ or $a_i = 1$ for $i = r, r+1, \cdots, s$; $a_s = 1$; and

$$N = \sum_{i=r}^{s} a_i U_i \; . \tag{1}$$

Definition 2: Given a representation of N of the form indicated above, we define the magnitude of the representation to be the sum of the coefficients a_i.

Definition 3: A representation of N in $\{U_n\}$ is said to be Minimal (or Maximal) if and only if the magnitude of the representation is less than or equal to (or greater than or equal to) the magnitude of every other representation of N in $\{U_n\}$.

This definition agrees with the intuitive notions of minimal and maximal representations in the sense that, for example, a minimal representation employs the fewest possible elements of the sequence $\{U_n\}$. Ferns, working with the special case $r = 2$ (the Fibonacci numbers) defined these ideas in a mathematically simpler but intuitively less satisfying way, which Mohanty generalized essentially as follows:

Definition 4: A representation of the form given by (1) in $\{U_n\}$ is minimal (or maximal) if and only if $a_i a_{i+j} = 0$ (or $a_i + a_{i+j} \geq 1$, respectively) for all $j = 1, 2, \cdots, r - 1$ and $i = r, r + 1, \cdots, s - j$.

It is easy to see that, for $r = 2$, these two definitions are equivalent. For if a representation in $\{F_n\}$ fails Definition 4, then, for some i, $a_i a_{i+1} = 1$ or $a_i + a_{i+1} = 0$ and the relation

$$F_{n+2} = F_{n+1} + F_n$$

can be applied to force conformity to Definition 4 and simultaneously to decrease (or increase) the magnitude of the representation, indicating that the original representation failed Definition 3 also. On the other hand, any representation not in accord with Definition 3 can be made to conform by suitable application of the relation cited above, which applications require the existence of coefficients a_i and a_{i+1} such that Definition 4 fails initially. Hence:

Theorem 1: If $r = 2$, then Definitions 3 and 4 are equivalent.

The main result of this paper is a proof (Theorems 2 and 3) of the converse of Theorem 1. It is clear that every positive integer N has at least one Minimal representation and one Maximal representation in $\{U_n\}$. Further, we have

Lemma 1: Every positive integer has a unique minimal representation in $\{U_n\}$ and a unique maximal representation in $\{U_n\}$.

Proof: This is established in [2], Lemmas 1 and 2.

Therefore, it suffices to display, for each value of r greater than 2, an integer whose minimal (or maximal) representation is not Minimal (or Maximal). Toward that end, we consider the triangular numbers.

Definition 5: The triangular numbers $\{T_n\}$ are given by

$$T_1 = 1; \qquad T_n = n + T_{n-1} .$$

If in the above definition n is allowed to take on successively the values m + 3, m + 4, and m + 5, and the resulting equations are summed, the following useful identity is obtained:

$$T_{m+5} = T_{m+2} + 3m + 12. \tag{2}$$

Lemma 2: If k is an integer such that $1 \leq k \leq r$, then:

$$U_k = 1 \tag{3}$$

$$U_{r+k} = k + 1 \tag{4}$$

$$U_{2r+k} = r + T_{k+1} \tag{5}$$

$$U_{3r+k} = r(k + 2) + T_r + T_1 + T_2 + \cdots + T_{k+1} . \tag{6}$$

Proof: These may be established by infinite induction.

Lemma 3: If $r \geq 6$ and $r = 3m$ for some integer m, then

$$U_{10m+1} + U_{6m+3} + U_{3m+1} = U_{10m} + U_{7m+4} .$$

Proof: Let $r = 3m$ in Equations (4), (5), and (6):

$$U_{3m+k} = k + 1 \tag{4'}$$

$$U_{6m+k} = 3m + T_{k+1} \tag{5'}$$

$$U_{9m+k} = 3m(k + 2) + T_{3m} + T_1 + T_2 + \cdots + T_{k+1} . \tag{6'}$$

Also,

$$(7)\quad U_{10m+1} + U_{6m+3} + U_{3m+1} - U_{10m} - U_{7m+4} = U_{9m+(m+1)} + U_{6m+(3)} + U_{3m+(1)} - U_{9m+(m)} - U_{6m+(m+4)} \,.$$

Since the parenthesized term in each subscript of (7) is less than or equal to $r = 3m > 6$ implies that $m + 4 \le r$, we can substitute Equations (4'), (5'), (6') in (7) appropriately with k equal to the term in parentheses:

$$\begin{aligned} &U_{10m+1} + U_{6m+3} + U_{3m+1} - U_{10m} - U_{7m+4} \\ &\quad = (3m(m+3) + T_{3m} + T_1 + T_2 + \cdots + T_{m+2}) + (T_4 + 3m) + (2) \\ &\quad\quad - (3m(m+2) + T_{3m} + T_1 + T_2 + \cdots + T_{m+1}) - (3m + T_{m+5}) \\ &\quad = T_{m+2} + 3m + 12 - T_{m+5} \,. \end{aligned}$$

In view of Equation (2), this establishes the Lemma.

Lemma 4: If $r \ge 6$ and $r = 3m + 1$ for some integer m, then

$$U_{10m+4} + U_{6m+5} + U_{3m+1} = U_{10m+3} + U_{7m+6} \,.$$

Lemma 5: If $r \ge 6$ and $r = 3m + 2$ for some integer m, then

$$U_{10m+8} + U_{6m+7} + U_{3m+4} = U_{10m+7} + U_{7m+9} \,.$$

Proof: Lemmas 4 and 5 are proved in a manner identical with that above, using Equations (2) through (6). Details are omitted.

Theorem 2: Given a sequence U_n satisfying Definition 1 with $r \ge 2$, there exists a positive integer N such that the unique minimal representation of N in U_n is not Minimal.

Proof: For $r \ge 6$, let

$$\begin{aligned} N &= U_{10m+1} + U_{6m+3} + U_{3m+1} \quad \text{if} \quad r = 3m, \\ N &= U_{10m+4} + U_{6m+5} + U_{3m+1} \quad \text{if} \quad r = 3m + 1, \end{aligned}$$

and

$$N = U_{10m+8} + U_{6m+7} + U_{3m+4} \quad \text{if} \quad r = 3m + 2\,.$$

The representation given for N is minimal, but in view of Lemmas 3, 4, and 5, is not Minimal. Similarly, let

$$N = 167 = U_{15} + U_{11} + U_8 + U_3 = U_{14} + U_{13} + U_{10} \quad \text{for} \quad r = 3,$$

$$N = 62 = U_{15} + U_{10} + U_5 = U_{14} + U_{13} \quad \text{for} \quad r = 4,$$

and let

$$N = 54 = U_{17} + U_{11} + U_5 = U_{16} + U_{14} \quad \text{for} \quad r = 5.$$

In each of these cases, the first expression for N is minimal but is obviously not Minimal. Thus counterexamples to the minimal-Minimal correlation have been exhibited for all sequences $\{U_n\}$ corresponding to $r > 2$; the proof is complete.

Theorem 3: Given a sequence $\{U_n\}$ satisfying Definition 1 with $r > 2$, there exists an integer N such that the unique maximal representation on N in $\{U_n\}$ is not Maximal.

Proof: For $r \geq 5$, direct substitution using Equations (4) and (5) serves to establish that

$$U_{2r+5} + \sum_{i=0}^{r+2} U_{r+i} = U_{2r+4} + U_{2r+3} + U_{2r+2} + \sum_{i=1}^{r} U_{r+1}\,.$$

Similarly, we can show that for $r = 4$,

$$U_{13} + U_{10} + U_9 + \sum_{i=3}^{7} U_i = U_{12} + U_{11} + U_{10} + U_8 + U_7 + U_5 + U_4$$

and for $r = 3$,

$$U_{14} + \sum_{i=4}^{11} U_i = U_{13} + U_{12} + U_{11} + U_{10} + U_8 + U_7 + U_6 + U_4 .$$

As in the proof of Theorem 2, each of these equations provides two representations for N: the first is maximal by Definition 4, but is of smaller magnitude than the second, and hence not Maximal. This is sufficient to establish the theorem.

Taken together, Theorems 1, 2 and 3 establish that every minimal representation is Minimal and every maximal representation is Maximal in $\{U_n\}$ if and only if $r = 2$, which was the promised result.

Mohanty noted in [2] that $\{U_n\}$ is a special case of the generalized Fibonacci numbers of V. C. Harris and Carolyn C. Styles [3]; specifically,

$$U_n = \sum_{i=0}^{[n/r]} \binom{n - i(r - 1)}{i} ,$$

where $[n/r]$ denotes the greatest integer in n/r. The Tribonacci numbers of Mark Feinberg [4], [5] can be defined as the sums of the rising diagonals of the trinomial triangle generated by $(1 + x + x^2)^n$, and can be generalized in an analogous manner. If the coefficient of x^k in the expansion of $(1+x+x^2)^n$ is denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_3$, then we can define the generalized tribonacci sequence $\{V_{r,n}\}$ by

$$V_{r,n} = \sum_{i=0}^{\infty} \begin{bmatrix} n - i(r - 1) \\ i \end{bmatrix}_3 .$$

As before, we assert without proof that $\{V_n\} = \{V_{r,n}\}$ is complete, even under the restriction that the first $r - 1$ elements of the sequence not to be employed in any integer representations. Further, we extend Definitions 3 and 4 to apply to the new family of sequences, and assert that Theorem 1 can be similarly generalized.

The following theorem is offered without proof.

Theorem 2': If $r \geq 4$, there exists a positive integer N such that the minimal representation of N in $\{V_n\}$ is not Minimal. Specifically,

$$N = V_{4r} + V_{2r+3} + V_{r+1} = V_{4r-1} + V_{3r+2} .$$

The left side is in proper form for a minimal but the right side has fewer digits. One can easily find an infinite number of other exceptions for each r. For example, add V_{5r+j} to each side for $j = 1, 2, 3, \cdots$.

One can secure a counterexample for the maximal which is not Maximal by subtracting each of those N's above from

$$\sum_{j=r}^{4r+1} V_j .$$

REFERENCES

1. H. H. Ferns, "On the Representation of Integers as Sums of Distinct Fibonacci Numbers," Fibonacci Quarterly, 3 (1965), pp. 21-30.
2. S. G. Mohanty, "On a Partition of Generalized Fibonacci Numbers," Fibonacci Quarterly, 6 (1968), pp. 22-33.
3. V. C. Harris and Carolyn C. Styles, "A Generalization of Fibonacci Numbers," Fibonacci Quarterly, 2 (1964), pp. 277-289.
4. Mark Feinberg, "Fibonacci-Tribonacci," Fibonacci Quarterly, 1 (1963), pp. 71-74.
5. Mark Feinberg, "New Slants," Fibonacci Quarterly, 2 (1964), pp. 223-227.

ERRATA

Please make the following change in the article by London and Finkelstein, "On Fibonacci and Lucas Numbers which are Perfect Powers," Dec. 1969, p. 481:

Equation (14) should read: $Y^2 - 500 = X^3$.