A LUCAS ANALOGUE

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The Lucas sequence is defined in terms of the roots r_1 and r_2 of the equation $x^2 = x + 1$ by the formula:

$$L_n = r_1^n + r_2^n \quad .$$

For this simple quadratic equation, the roots can be calculated as:

$$r_1 = \frac{1 + \sqrt{5}}{2}, \quad r_2 = \frac{1 - \sqrt{5}}{2} \quad .$$

It can be ascertained directly that $r_1 + r_2 = 1$ and $r_1^2 + r_2^2 = 3$. Then, since

$$r_1^{n+1} = r_1^n + r_1^{n-1}$$

and

$$r_2^{n+1} = r_2^n + r_2^{n-1} ,$$

it follows by mathematical induction that if:

$$L_{n-1} = r_1^{n-1} + r_2^{n-1}$$

and

$$L_n = r_1^n + r_2^n ,$$

then

$$L_{n+1} = r_1^{n+1} + r_2^{n+1} ,$$

by direct addition of the two equations.

If we seek to extend this idea to a sequence in which the last three consecutive terms are added to get the next term, the corresponding equation to be used is: $x^3 = x^2 + x + 1$, which has three roots r_1, r_2, r_3. These need not be calculated. It suffices to know that:

$$\sum_{i=1}^{3} r_i = 1, \quad {\sum_{i,j=1}^{3}}' r_i r_j = -1 \quad \text{and} \quad r_1 r_2 r_3 = 1 ,$$

using the standard relations between roots and coefficients of an equation. The analogous definition of a sequence in terms of the roots would be:

$$T_n = r_1^n + r_2^n + r_3^n .$$

The first three values calculated on the basis of symmetric function formulas would be:

$$T_1 = \sum_{i=1}^{3} r_i = 1$$

$$T_2 = \sum_{i=1}^{3} r_i^2 = (\Sigma r_i)^2 - 2\Sigma r_i r_j = 3$$

$$T_3 = \sum r_i^3 = (\Sigma r_i)^3 - 3(\Sigma r_i)(\Sigma r_i r_j) + 3 r_1 r_2 r_3$$

$$T_3 = 1 + 3 + 3 = 7 .$$

With these values defined, T_4 would be equal to

$$\Sigma r_i^4$$

on the basis of the relation

$$r_i^4 = r_i^3 + r_i^2 + r_i .$$

Carrying the procedure one more step, if the last four quantities in a sequence are added to obtain the next quantity, the roots of the equation

$$x^4 = x^3 + x^2 + x + 1$$

would be employed with the sequence defined by:

$$Q_n = \sum_{i=1}^{4} r_i^n .$$

With the aid of symmetric functions, it can be shown that:

$$Q_1 = 1, \quad Q_2 = 3, \quad Q_3 = 7, \quad Q_4 = 15.$$

From these preliminary investigations, two points emerge: (1) As we extend the Lucas analogue, the basic starting quantities carry over from one stage to the next; (2) All the initial quantities are of the form $2^k - 1$. That these relations remain generally true is not difficult to prove.

Consider the n^{th} order recursion relation:

$$x^n = x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1 ,$$

which corresponds to adding the last n quantities to obtain the next quantity in a sequence. Expressed as a polynomial equated to zero, this becomes:

$$x^n - x^{n-1} - x^{n-2} - x^{n-3} - \cdots - x - 1 = 0 ,$$

so that regardless of the degree of the equation:

$$\Sigma r_i = 1, \quad \Sigma r_i r_j = -1, \quad \Sigma r_i r_j r_k = 1, \quad \text{etc.}$$

Then, since

$$\Sigma r_i^j$$

is expressible in terms of those summations which have j or less components, it follows that the sum must be the same for any two equations having a degree greater than or equal to j. This accounts for the persistence of the basic starting quantities from one stage to the next.

Now assume that for the equation of degree n - 1

$$\Sigma r_i^j = 2^j - 1 \qquad (j = 1, 2, \cdots, n - 1) .$$

By the equation for the n^{th} degree:

$$\begin{aligned}\Sigma r_i^n &= \Sigma r_i^{n-1} + \Sigma r_i^{n-2} + \Sigma r_i^{n-3} + \cdots + \Sigma r_i + \Sigma 1 \\ &= (2^{n-1} - 1) + (2^{n-2} - 1) + (2^{n-3} - 1) \cdots (2 - 1) + n \\ &= \sum_{i=0}^{n-1} 2^k = 2^n - 1 .\end{aligned}$$

This proves the second contention.

CONCLUSION

For an n^{th} order recursion relation of the form

$$T_{m+n+1} = T_{m+n} + T_{m+n+1} + T_{m+n-2} + \cdots + T_{m+1} ,$$

it is possible to define a Lucas-type sequence using the roots of the equation $x^n = x^{n-1} + x^{n-2} + x^{n-2} + \cdots + x + 1$ with

$$T_m = \Sigma r_i^m .$$

With such a definition, the first n starting values would be given by:

$$T_k = 2^k - 1 \qquad (k = 1, 2, \cdots, n) .$$