ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-190 A repeat of B-186 with a typographical error corrected.

Let L_n be the n^{th} Lucas number and show that

$$L_{5n}/L_n = [L_{2n} - 3(-1)^n]^2 + (-1)^n 25F_n^2 .$$

B-191 Proposed by Guy A. Guillottee, Montreal, Quebec, Canada.

In this alphametic, each letter represents a particular but different digit, all ten digits being represented here. It must only be that well-known mathematical teaser from Toronto, J. A. H. Hunter, but what is the value of HUNTER?

$$\begin{array}{r} MR \\ HUNTER \\ MADE \\ A \\ \hline TEASER \end{array}$$

B-192 Proposed by Warren Cheves, Littleton, North Carolina.

Prove that $F_{3n} = L_n F_{2n} - (-1)^n F_n$.

B-193 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that $L_{n+p} \pm L_{n-p}$ is $5F_pF_n$ or L_pL_n depending on the choice of sign on whether p is even or odd.

B-194 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Show that $L_{n+4k} - L_n = 5F_k[F_{n+3k} + (-1)^n F_{n+k}]$.

B-195 Proposed by David Zeitlin, Minneapolis, Minnesota.

Let $\left\{ {n \atop r} \right\}$ denote $L_nL_{n-1} \cdots L_{n-r+1}/L_1L_2 \cdots L_r$. Show that

$$L_n^3/6 = \left\{ {n+2 \atop 3} \right\} - 2\left\{ {n+1 \atop 3} \right\} - \left\{ {n \atop 3} \right\}.$$

Following is a list of solvers whose names were inadvertently omitted from lists in recent issues:

B-144 Don Allen

B-148, B-149, B-150, B-151, B-153 — D. V. Jaiswal

B-160, B-161, B-163 — H. V. Krishna

B-166 Michael Yoder

B-167 T. J. Cullen, Bruce W. King, R. W. Sielaff, Michael Yoder

B-168 Michael Yoder

B-169 Wray G. Brady, Michael Yoder

B-170, B-171 — Michael Yoder

SOLUTIONS

A CUBIC IDENTITY

B-172 Proposed by Gloria C. Padilla, Albuquerque High School, Albuquerque, New Mexico.

Let $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$ for $n = 0, 1, \cdots$. Show that

$$F_{n+2}^3 = F_n^3 + F_{n+1}^3 + 3F_nF_{n+1}F_{n+2}.$$

Solution by C. B. A. Peck, State College, Pennsylvania.

We obtain the desired result from $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$ on substituting $F_n = x$ and $F_{n+1} = y$ and hence $F_{n+2} = x + y$. Note that the result does not depend on the initial values

Also solved by W. C. Barley, Wray G. Brady, T. J. Cullen, Andrew Dias, David Englund, Herta T. Freitag, Bernard G. Hoerbelt, John E. Homer, Jr., John Ivie, Bruce W. King, Peter A. Lindstrom, Bruce Lynn, John W. Milsom, Klaus-Günther Recke, Michael Rennie, Gerald Satlow, A. G. Shannon, Richard W. Sielaff, Charles W. Trigg, John Wessner, Gregory Wulczyn, Michael Yoder, and the Proposer.

ANOTHER CUBIC IDENTITY

B-173 Proposed by Gloria C. Padilla, Albuquerque High School, Albuquerque, New Mexico.

Show that

$$F_{3n} = F_{n+2}^3 - F_{n-1}^3 - 3F_n F_{n+1} F_{n+2} \,.$$

Solution by T. J. Cullen, California State Polytechnic College, Pomona, California.

From Formula XXI on p. 68 of Vol. 1, Fibonacci Quarterly:

$$F_{3n} = F_n^3 + F_{n+1}^3 - F_{n-1}^3 \,.$$

Hence, by B-172,

$$F_{3n} = F_{n+2}^3 - F_{n-1}^3 - 3F_n F_{n+1} F_{n+2} \,.$$

Also solved by W. C. Barley, Wray G. Brady, Herta T. Freitag, Bernard G. Hoerbelt, John E. Homer, Jr., John Ivie, Bruce W. King, Peter A. Lindstrom, John W. Milsom, C. B. A. Peck, Klaus-Günther Recke, A. G. Shannon, Charles W. Trigg, Gregory Wulczyn, Michael Yoder, David Zeitlin, and the Proposer.

MODULO 10

B-174 Proposed by Mel Most, Ridgefield Park, New Jersey.

Let a be a non-negative integer. Show that in the sequence

$$2F_{a+1},\quad 2^2F_{a+2},\quad 2^3F_{a+3},\cdots$$

all differences between successive terms must end in the same digit.

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, New Mexico.

The sequence satisfies the recurrence

$$S_{n+2} = 2S_{n+1} + 4S_n \qquad (S_n = 2^nF_{a+n}) .$$

Thus

$$S_{n+2} - S_{n+1} = (S_{n+1} - S_n) + 5S_n \equiv S_{n+1} - S_n \pmod{10} ,$$

since all S_n are even.

Also solved by T. J. Cullen, David Englund, Hertá T. Freitag, C. B. A. Peck, Klaus-Günther Recke, A. G. Shannon, Charles W. Trigg, John Wessner, David Zeitlin, and the proposer.

A GENERALIZED 2-BY-2 DETERMINANT

B-175 Composed from the Solution by David Zeitlin to B-155.

Let r and q be constants and let $U_0 = 0$, $U_1 = 1$, $U_{n+2} = rU_{n+1} - qU_n$. Show that

$$U_{n+a}U_{n+b} - U_{n+a+b}U_n = q^nU_aU_b .$$

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, New Mexico.

For $a = 0$, the identity is obviously true; and if it is true for $a = 1$, it will be true by induction for all a. Similarly, the identity need only be proved for $b = 1$, and the problem reduces to that of proving $U_{n+1}^2 - U_{n+2}U_n = q^n$. For $n = 0$, this is true; and since

$$\begin{aligned} U_{n+2}^2 - U_{n+3}U_{n+1} &= U_{n+2}(rU_{n+1} - qU_n) - (rU_{n+2} - qU_{n+1})U_{n+1} \\ &= q(U_{n+1}^2 - U_{n+2}U_n) , \end{aligned}$$

the identity is verified for all a, b, and n.

Also solved by Wray G. Brady, T. J. Cullen, Herta T. Freitag, C. B. A. Peck, Klaus-Günther Recke, A. G. Shannon, Gregory Wulczyn, and David Zeitlin.

CUBES IN TERMS OF FIBONOMIALS ON DIAGONAL 3

B-176 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let $\begin{bmatrix} n \\ r \end{bmatrix}$ denote the Fibonomial Coefficient

$$F_n F_{n-1} \cdots F_{n-r+1} / F_1 F_2 \cdots F_r .$$

Show that

$$F_n^3 = \begin{bmatrix} n+2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} n+1 \\ 3 \end{bmatrix} - \begin{bmatrix} n \\ 3 \end{bmatrix} .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Let H_n satisfy $H_{n+2} = H_{n+1} + H_n$, and define

$$\begin{Bmatrix} n \\ r \end{Bmatrix} = H_n H_{n-1} \cdots H_{n-r+1} / H_1 H_2 \cdots H_r ,$$

where $H_i > 0$, $i = 1, 2, \cdots$. Since

(1) $$2H_n^2 = H_{n+2}H_{n+1} - 2H_{n+1}H_{n-1} - H_{n-1}H_{n-2} ,$$

multiplication of (1) by H_n gives

(2) $$2H_n^3 = H_1 H_2 H_3 \left(\begin{Bmatrix} n+2 \\ 3 \end{Bmatrix} - 2\begin{Bmatrix} n+1 \\ 3 \end{Bmatrix} - \begin{Bmatrix} n \\ 3 \end{Bmatrix} \right) .$$

The desired result is obtained from (2) for

$$H_n = F_n, \quad \text{where} \quad \left\{ {n \atop r} \right\} = \left[{n \atop r} \right].$$

Also solved by L. Carlitz, T. J. Cullen, Herta T. Freitag, John E. Homer, Jr., Peter A. Lindstrom, John W. Milsom, C. B. A. Peck, Klaus-Günther Recke, A. G. Shannon, Charles W. Trigg, Gregory Wulczyn, Michael Yoder, and the Proposer.

FOURTH POWERS IN TERMS OF FIBONOMIALS

B-177 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Using the notation of B-176, show that

$$F_n^4 = \left[{n+3 \atop 4} \right] - a\left[{n+2 \atop 4} \right] - a\left[{n+1 \atop 4} \right] + \left[{n \atop 4} \right],$$

for some integer a and find a.

Solution by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Letting $n = 2$, we find that a would have to be 4. Then letting $a = 4$, both sides satisfy the same fourth-order (i.e., five-term) recurrence relation. Hence it suffices to verify the formula for $n = 0, 1, 2, 3$ and it follows for all values of n by induction.

Also solved by L. Carlitz, T. J. Cullen, Herta T. Freitag, John E. Homer, Jr., C. B. A. Peck, Klaus-Günther Recke, Charles W. Trigg, Gregory Wulczyn, Michael Yoder, David Zeitlin, and the Proposer.

[Continued from p. 438.]

It is found, also, that the slope m of the distances z_n versus the Fibonacci Series f_n for each planet system is a power law function of the mass M and radius R of the planet in the form

$$m \propto M^3R^{-7}.$$