THE SMALLEST NUMBER WITH DIVISORS A PRODUCT OF DISTINCT PRIMES

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1. INTRODUCTION

Let P_i denote the i^{th} prime. This paper contains a proof that there is a number k such that for $k > K$, the number

$$P_1^{P_{k+s}-1} P_2^{P_{k-1+s}-1} \cdots P_r^{P_{k-r+1+s}-1} \cdots P_k^{P_{1+s}-1}$$

is the smallest number having

$$P_{k+s} P_{k-1+s} \cdots P_{1+s}$$

divisors, where $s \geq 0$, $1 \leq r \leq k - 1$.

2. LEMMAS

The following Lemma is repeatedly used in the proof of Lemma 2.

Lemma 1. There exist positive constants

$$C = \frac{1}{9 \log 2}$$

and d such that $cr \log r < P_r < dr \log r$. See [2 p. 186].

Lemma 2. Let P_i denote the i^{th} prime. There exists a number K large enough such that for $k > K$, we have

$$P_r^{P_{k-r+1}} > P_k \ ,$$

for $r = 1, 2, \cdots, k - 1$.

Proof. For $r = 1$, we do have

$$2^{P_k} > P_k \tag{1}$$

for all k.

For $r = 2$, by Lemma 1, we have

$$3^{P_{k-1}} > 3^{c(k-1)\log(k-1)} .$$

There exists a k_2 such that for $k > k_2$, we have

$$3^{P_{k-1}} > 3^{c(k-1)\log(k-1)} > dk \log k \quad . \tag{2}$$

By Lemma 1 and Eq. (2), there is a constant k_2 such that for $k > k_2$, we have

$$3^{P_{k-1}} > P_k \quad . \tag{3}$$

Similarly, for

$$3 \le r \le \frac{k+1}{2} \quad ,$$

there is a constant k_r such that for $k > k_r$, we have

$$P_r^{P_{k-r+1}} > P_k \quad . \tag{4}$$

For $r = k - 1$, by Lemma 1, we have

$$\begin{aligned} P_{k-1}^2 &> (c(k-1)\log(k-1))^2 \\ &= c^2(k-1)^2 \log^2(k-1) \quad . \end{aligned}$$

Hence, there is a constant k_{k-1} such that for $k > k_{k-1}$, we have

$$P_{k-1}^2 > c^2(k-1)^2\log^2(k-1) > dk\log k > P_k \, . \tag{5}$$

Similarly, for

$$\frac{k+1}{2} < r \le k - 2,$$

there is a constant k_r such that

$$\begin{aligned} P_r^{P_{k-r+1}} &> (cr \log r)^{P_{k-r+1}} \\ &> dk \log k \\ &> P_k \end{aligned}$$

for $k > k_r$.

Let K be the maximum of $k_1, k_2, \cdots, k_{k-1}$. Then for $k > K$, we have

$$P_r^{P_{k-r+1}} > P_k \quad (r = 1, 2, \cdots, k-1) \text{ Q.E.P.}$$

Immediately following from Lemma 2, we have

Lemma 3. There is a constant K such that for $k > K$, we have

$$P_r^{P_{k-r+1+s}} > P_k > P_i,$$

for $r = 1, 2, \cdots, k-1$, and $s \ge 0$, where $r < i \le k-1$.

Since we know $P_1^{AB-1} > P_1^{A-1}P_2^{B-1}$ if $A > 1$, $B > 1$, and $P_1^A > P_2$ ([1, Lemma 1]), together with Lemma 3, we conclude the following theorem.

Theorem. There is a constant K such that for $k > K$,

$$P_1^{P_{k+s}-1} P_2^{P_{k-1+s}-1} \cdots P_r^{P_{k-r+1+s}-1} \cdots P_k^{P_{1+s}-1}$$

is the smallest number such that it has $P_{k+s}P_{k-1+s} \cdots P_{k-r+1+s} \cdots P_{1+s}$ divisors.

REFERENCES

1. M. E. Grost, "The Smallest Number with Given Number of Divisors," American Mathematical Monthly, Vol. 75, No. 7, 1968.
2. Ivan Niven and Herbert S. Zuckerman, An Introduction to the Theory of Numbers, John Wiley and Sons, Inc., Second Edition, 1966.