A THEOREM CONCERNING ODD PERFECT NUMBERS

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1. INTRODUCTION

Although the question of the existence of odd perfect numbers is still open, many necessary conditions for an odd integer to be perfect have been established. The oldest of these is due to Euler (see p. 19 in [1]), who proved that if n is an odd perfect number then $n = p^a k^2$ where p is a prime, $k > 1$, $(p,k) = 1$, and $p \equiv a \equiv 1 \pmod 4$. In 1953, Touchard [6] proved that if n is odd and perfect, then either $n = 12t + 1$ or $n = 36t + 9$. More recently the first author [5] has established upper and lower bounds for

$$\sum_{p|n} \frac{1}{p}$$

where n is an odd perfect number. In fact, these bounds are improved ones over those established in [3] and [4]. For convenience, we give in Table 1 the results of [5] correct to five decimal places.

Table 1

	Lower Bound	Upper Bound	Range
(A)	.64412	.67841	.03429
(B)	.65696	.69315	.03619
(C)	.59595	.67377	.07782
(D)	.59993	.66172	.06179

Our objective in the present paper is to improve (some of) the results of [5]. Our bounds for

$$\sum_{p|n} \frac{1}{p}$$

are given in Theorem 1 while the five decimal place approximations appear in Table 2. In what follows, n denotes an odd perfect number, and p denotes a prime. The notation

$$\sum_{p=5}^{11} \frac{1}{p} ,$$

for example, will be used to represent the sum

$$\frac{1}{5} + \frac{1}{7} + \frac{1}{11} .$$

Table 2

	Lower Bound	Upper Bound	Range
(A)	.64738	.67804	.03066
(B)	.66745	.69315	.02570
(C)	.59606	.67377	.07771
(D)	.60383	.65731	.05348

<u>Theorem 1.</u> If n is an odd perfect number, then

(A) if $n = 12t + 1$ and $5 \mid n$,

$$\sum_{p=5}^{19} \frac{1}{p} + \frac{\log \left\{ 2 \prod_{p=5}^{19} (p-1)/p \right\}}{23 \log (23/22)} < \sum_{p \mid n} \frac{1}{p} < \frac{1}{5} + \log (50/31) \;:$$

(B) if $n = 12t + 1$ and $5 \nmid n$,

$$\sum_{p=7}^{59} \frac{1}{p} + \frac{\log \left\{ 2 \prod_{p=7}^{59} (p-1)/p \right\}}{61 \log (61/60)} < \sum_{p \mid n} \frac{1}{p} < \log 2 \;;$$

.

(C) if $n = 36t + 9$ and $5 \mid n$,

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{17} + \frac{\log(256/255)}{257\log(257/256)} < \sum_{p|n} \frac{1}{p} < \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \log(65/61) \ ;$$

(D) if $n = 36t + 9$ and $5 \nmid n$,

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{\log(80/77)}{13\log(13/12)} < \sum_{p|n} \frac{1}{p} < \frac{1}{3} + \frac{1}{13} + \frac{1}{17} + \log(37349/30941) \ .$$

The upper bounds in (B) and (C) are due to the first author [5]. The rest of the theorem is new.

2. THE UPPER BOUNDS

In this section, we shall establish the upper bounds for

$$\sum_{p|n} \frac{1}{p}$$

given in (A) and (D) of Theorem 1. Our argument parallels that in [5].

According to Euler's theorem, we can write

$$n = p_0^{a_0} p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \ ,$$

where $p_0 \equiv a_0 \equiv 1 \pmod 4$ and $a_j \equiv 0 \pmod 2$ for $1 \leqq j \leqq k$. We assume that $p_1 < p_2 < \cdots < p_k$. Since n is perfect, we have immediately

$$(2.0) \qquad 2 = \prod_{j=0}^{k} 1 - 1/p_j^{a_j+1} \prod_{j=0}^{k} (1 - 1/p_j)^{-1} \ ,$$

so that

$$\log 2 = \sum_{j=0}^{k} \frac{1}{p_j} + \sum_{j=1}^{k} \sum_{i=1}^{\infty} 1/(i+1)p_j^{i+1} - 1/ip_j^{(a_j+1)i} \tag{2.1}$$

$$+ 1/2p_0^2 - 1/p_0^{a_0+1} + \sum_{i=2}^{\infty} 1/(i+1)p_0^{i+1} - 1/ip_0^{(a_0+1)i} .$$

<u>Remark 1.</u> Since $a_j \geqq 2$ for $1 \leqq j \leqq k$ each term is positive in the second summation of (2.1).

<u>Remark 2.</u> Since $a_0 \geqq 1$ and $i \geqq 2$, each term is positive in the last summation of (2.1).

<u>Remark 3.</u> Since $a_0 \geqq 1$, we have

$$1/2p_0^2 - 1/p_0^{a_0+1} \geqq -1/2p_0^2 .$$

<u>Remark 4.</u> Since a_0 is odd $(p_0 + 1) | \sigma(p_0^{a_0})$, and since $n = \sigma(n)/2$, it follows that $(p_0 + 1)/2 | n$ and <u>a fortiori</u> that n is divisible by a prime $p_s \leqq (p_0 + 1)/2$.

<u>Remark 5.</u> If p_s is the prime mentioned in Remark 4 and $p_0 > 5$, then

$$W = 1/2p_s^2 - 1/p_s^{a_s+1} + 1/2p_0^2 = 1/p_0^{a_0+1} > 0 .$$

For since $3 \leqq p_s \leqq (p_0 + 1)/2$, $a_s \geqq 2$, $a_0 \geqq 1$, we have

$$W \geqq 1/2p_s^2 - 1/3p_s^2 - 1/2p_0^2 \geqq 2/3(p_0 + 1)^2 - 1/2p_0^2 > 0 .$$

We consider first the case $n = 12t + 1$ and $5|n$. Since $3 \nmid n$, we see from Remark 4 that $p_0 \neq 5$. Therefore, $p_1 = 5$.

If $(p_0 + 1)/2 \neq 5^m$, then we can assume that the p_s of Remark 4 is <u>not</u> 5. Since $a_1 \geqq 2$, it follows from (2.1) and Remarks 1, 2, 5 that

$$\log 2 > \sum_{p|n} \frac{1}{p} + \sum_{i=1}^{\infty} 1/(i+1)5^{i+1} - 1/i5^{3i}$$

$$= \sum_{p|n} \frac{1}{p} - \frac{1}{5} - \log(1 - 1/5) + \log(1 - 1/5^3) .$$

Therefore,

$$\sum_{p|n} \frac{1}{p} < \frac{1}{5} + \log (50/31) .$$

Since the smallest prime such that $(p + 1)/2 = 5^m$ is $p = 1249 = 2 \cdot 5^4 - 1$, we see that if $(p_0 + 1)/2 = 5^m$, then $p_0 \geqq 1249$, so that $-1/2p_0^2 \geqq -1/2(1249)^2$. Also, in this case, it follows from Remark 4 that $a_1 \geqq 4$. From (2.1) and Remarks 1, 2, 3, we have

$$\log 2 > \sum_{p|n} \frac{1}{p} - \frac{1}{5} - \log (1 - 1/5) + \log(1 - 1/5^5) - 1/2(1249)^2 .$$

Therefore,

$$\sum_{p|n} \frac{1}{p} < \frac{1}{5} + 1/2(1249)^2 + \log (1250/781) < \frac{1}{5} + \log (50/31) .$$

This completes the discussion of the upper bound for this case. We remark that the upper bound established in [5] for (A) exceeds ours by 1/2738.

Turning to the case $n = 36t + 9$ and $5 \nmid n$ we have $p_1 = 3$. We consider four mutually exclusive and exhaustive possibilities.

First, suppose that $a_1 = 2$ and $p_0 = 17$. Since $\sigma(3^2) = 13$ and $n = \sigma(n)/2$, we see that $13|n$. Let $13 = p_r$. If $a_r = 2$ then since $\sigma(13^2) = 183$ and since $p_0 + 1 = 18$, it would follow from Remark 4 that $3^3|n$. Since this is impossible, we conclude that $a_r \geqq 4$. Since $a_0 \geqq 1$, it follows from (2.1) and Remark 1 that

$$\log 2 > \sum_{p|n} \frac{1}{p} - \frac{1}{3} - \log (1 - 1/3) + \log (1 - 1/3^3) - \frac{1}{13} - \log (1 - 1/13)$$

$$+ \log (1 - 1/13^5) - \frac{1}{17} - \log (1 - 1/17) + \log (1 - 17^2) .$$

Therefore,

$$\sum_{p|n} \frac{1}{p} < \frac{1}{3} + \frac{1}{13} + \frac{1}{17} + \log (37349/30941) .$$

Second, suppose that $a_1 = 2$ and $p_0 = 13$. Then the p_s of Remark 4 is 7, and it follows from (2.1) and Remark 1 that

$$\log 2 > \sum_{p|n} \frac{1}{p} - \frac{1}{3} - \log (1 - 1/3) + \log (1 - 1/3^3) - \frac{1}{7} - \log (1 - 1/7)$$

$$+ \log (1 - 1/7^3) - \frac{1}{13} - \log (1 - 1/13) + \log (1 - 1/13^2).$$

Therefore,

$$\sum_{p|n} \frac{1}{p} < \frac{1}{3} + \frac{1}{7} + \frac{1}{13} + \log (21/19) < \frac{1}{3} \quad \frac{1}{13} \quad \frac{1}{17} + \log (37349/30941) .$$

Next, suppose that $a_1 = 2$ and $p_0 > 17$. As before, we have $13|n$, while $p_0 \geqq 37$. For if $p_0 = 29$, it would follow from Remark 4 that $5|n$ which is impossible. From (2.1) and Remarks 1, 2, 3, we have

$$\log 2 > \sum_{p|n} \frac{1}{p} - \frac{1}{3} - \log (1 - 1/3) + \log (1 - 1/3^3) - \frac{1}{13}$$

$$- \log (1 - 1/13) + \log (1 - 1/13^3) - 1/2(37)^2 .$$

Therefore,

$$\sum_{p|n} \frac{1}{p} < \frac{1}{3} + \frac{1}{13} + \frac{1}{2738} + \log (78/61) < \frac{1}{3} + \frac{1}{13} + \frac{1}{17} + \log (37349/30941) .$$

Finally, suppose that $a_1 \geqq 4$. Since $p_0 \geqq 13$, we have $-1/2p_0^2 \geqq -1/338$. From (2.1) and Remarks 1, 2, 3, it follows that

$$\log 2 > \sum_{p|n} \frac{1}{p} - \frac{1}{3} - \log(1 - 1/3) + \log(1 - 1/3^5) - \frac{1}{338} .$$

Therefore,

$$\sum_{p|n} \frac{1}{p} < \frac{1}{3} + \frac{1}{338} + \log(162/121) < \frac{1}{3} + \frac{1}{13} + \frac{1}{17} + \log(37349/30941) .$$

This completes the discussion of the upper bound for this case.

3. THE LOWER BOUNDS

In this section, we change our notation and write simply

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} ,$$

where $p_1 < p_2 < \cdots < p_k$. We first establish two lemmas.

Lemma 1. If

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

is an odd perfect number and q is a prime such that $p_N < q \leqq p_{N+1}$, then

$$\log 2 < \log\left\{\prod_{j=1}^{N} p_j/(p_j - 1)\right\} + q \log \frac{q}{q-1}\left(\sum_{p|n} \frac{1}{p} - \sum_{j=1}^{N} \frac{1}{p_j}\right) .$$

Proof. From (2.0), it follows that

$$2 < \prod_{p|n} (1 - 1/p)^{-1} .$$

Taking logarithms, we have

$$\begin{aligned}
\log 2 &< \log \prod_{j=1}^{N} p_j/(p_j - 1) + \sum_{j=N+1}^{k} \sum_{i=1}^{\infty} 1/ip_j^i \\
&\leqq \log \quad {}_{j=1}^{N} p_j/(p_j - 1 + \sum_{j=N+1}^{k} \sum_{i=1}^{\infty} 1/(ip_j q^{i-1}) \\
&= \log \prod_{j=1}^{N} p_j/(p_j - 1) + \sum_{j=N+1}^{k} \frac{1}{p_j} \sum_{i=1}^{\infty} q/iq^i \\
&= \log \prod_{j=1}^{N} p_j/(p_j - 1) + q \log \frac{q}{q-1} \left(\sum_{p|n} \frac{1}{p} - \sum_{j=1}^{N} \frac{1}{p_j} \right) .
\end{aligned}$$

The necessary modifications in both the statement and proof of this lemma in case $q \leqq p_1$ or $p_k < q$ are obvious and are therefore omitted.

Lemma 2. The function $f(x) = x \log x/(x - 1)$ is monotonic decreasing on the interval $[2,\infty)$.

Proof. We easily verify that

$$f'(x) = \log \left(1 + \frac{1}{x-1}\right) - \frac{1}{x-1} .$$

Since $\log(1 + z) < z$ if $0 < z \leqq 1$, we see immediately that $f'(x) < 0$ if $x \geqq 2$.

We are now prepared to prove the lower bounds for

$$\sum_{p|n} \frac{1}{p}$$

stated in Theorem 1. We shall defer the proof of (C) until last since it differs in spirit from the others.

From Lemma 1, we have

$$\sum_{p|n} \frac{1}{p} > \sum_{j=1}^{N} \frac{1}{p_j} + \frac{\log\left\{2 \prod_{j=1}^{N} (p_j - 1)/p_j\right\}}{q \log\{q/(q-1)\}} \tag{3.0}$$

while from Lemma 2 it follows easily that if s is a prime such that $s < q$ then

$$\frac{1}{s} + \frac{\log\{(s-1)/s\}}{q \log\{q/(q-1)\}} < 0 . \tag{3.1}$$

If $n = 12t + 1$ and $5|n$, then $p_1 = 5$. If r is the greatest prime less than q then it follows from (3.0) and (3.1) that

$$\sum_{p|n} \frac{1}{p} > \sum_{p=5}^{r} \frac{1}{p} + \frac{\log\left\{2 \prod_{p=5}^{r} (p-1)/p\right\}}{q \log\{q/(q-1)\}} . \tag{3.2}$$

An hour's work on a desk calculator shows that the right-hand member of (3.2) is maximal for $q = 23$, $r = 19$. This completes the proof for this case. We remark that the lower bound for (A) established in [5] is (3.2) with $q = 11$, $r = 7$.

If $n = 12t + 1$ and $5 \nmid n$, then $p_1 \geqq 7$. With r defined as before, it follows from (3.0) and (3.1) that

$$\sum_{p|n} \frac{1}{p} > \sum_{p=7}^{r} \frac{1}{p} + \frac{\log\left\{2 \prod_{p=7}^{r} (p-1)/p\right\}}{q \log\{q/(q-1)\}} . \tag{3.3}$$

Some rather tedious calculations verify that the right-hand member of (3.3) is maximal for $q = 61$, $r = 59$. The lower bound for (B) established in [5] is (3.3) with $q = 11$, $r = 7$.

If $n = 36t + 9$ and $5 \nmid n$, then $p_1 = 3$ and $p_2 \geqq 7$. With r defined as before, we have from (3.0) and (3.1),

$$\sum_{p|n} \frac{1}{p} > \sum_{p=3}^{r}{}^{*} \frac{1}{p} + \frac{\log\left\{2 \prod_{p=7}^{r} {}^{*} (p-1)/p\right\}}{q \log\left\{q/(q-1)\right\}} \tag{3.4}$$

where the asterisk indicates that the prime 5 is to be omitted from consideration. A few minutes of calculation verifies that the right-hand member of (3.4) is maximal for $q = 13$, $r = 11$. The lower bound for (D) established in [5] is (3.4) with $q = 7$, $r = 3$.

Now suppose that $n = 36t + 9$ and $5|n$. Then $7 \nmid n$ by a result of Kuhnel [2]. We consider three mutually exclusive and exhaustive possibilities.

If either 11 or 13 divides n, then

$$\sum_{p|n} \frac{1}{p} > \frac{1}{3} + \frac{1}{5} + \frac{1}{13} > \frac{1}{3} + \frac{1}{5} + \frac{1}{17} + \frac{\log(256/255)}{257 \log(257/256)} .$$

If neither 11 nor 13 divides n but $17|n$, then $p_3 = 17$ and either (i) $p_4 < 251$, or (ii) $p_4 > 251$. In case (i), we have

$$\sum_{p|n} \frac{1}{p} > \frac{1}{3} + \frac{1}{5} + \frac{1}{17} + \frac{1}{251} > \frac{1}{3} + \frac{1}{5} + \frac{1}{17} + \frac{\log(256/255)}{257 \log(257/256)}$$

In case (ii), if we take $q = 257$ in (3.0), we have

$$\sum_{p|n} \frac{1}{p} > \frac{1}{3} + \frac{1}{5} + \frac{1}{17} + \frac{\log(256/255)}{257 \log(257/256)} .$$

[Continued on p. 374.]