ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-178 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$a_{m,n} = \binom{m+n}{m}^2 .$$

Show that $a_{m,n}$ satisfies no recurrence of the type

$$\sum_{j=0}^{r} \sum_{k=0}^{s} c_{j,k} a_{m-j,n-k} = 0 \qquad (m \geq r, \quad n \geq s) ,$$

where the $c_{j,k}$ and r,s are all independent of m,n.

Show also that $a_{m,n}$ satisfies no recurrence of the type

$$\sum_{j=0}^{r} \sum_{k=0}^{n} c_{j,k} a_{m-j,n-k} = 0 \qquad (m \geq r, \quad n \geq 0) ,$$

where the $c_{j,k}$ and r are independent of m,n.

H-179 Proposed by D. Singmaster, Bedford College, University of London, London, England.

Let k numbers $p_1, p_2, \cdots, p_k$ be given. Set $\alpha_n = 0$ for $n < 0$; $\alpha_0 = 1$ and define α_n by the recursion

$$\alpha_n = \sum_{i=1}^{n} p_i \alpha_{n-i} \quad \text{for} \quad n > 0.$$

1. Find simple necessary and sufficient conditions on the p_i for

$$\lim_{n \to \infty} \alpha_n$$

to exist and be: (a) finite and nonzero; (b) zero; (c) infinite.

2. Are the conditions: $p_i \geq 0$ for $i = 1, 2, \cdots$, $p_i > 0$ and

$$\sum_{i=1}^{n} p_i = 1$$

sufficient for $\lim_{n \to \infty} \alpha_n$ to exist, be finite, and be nonzero?

H-180 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{k=0}^{n} \binom{n}{k}^3 F_k = \sum_{2k \leq n} \frac{(n+k)!}{(k!)^3 (n-2k)!} F_{(2n-3k)}$$

$$\sum_{k=0}^{n} \binom{n}{k}^3 L_k = \sum_{2k \leq n} \frac{(n+k)!}{(k!)^3 (n-2k)!} L_{(2n-3k)}$$

where F_k and L_k denote the k^{th} Fibonacci and Lucas numbers, respectively.

SOLUTIONS

SUMMARILY PRODUCTIVE

H-156 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Prove the identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(q)_n} \prod_{k=1}^{\infty} (1 - q^k) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q)_{2k}} z^{-k}$$

$$- \sum_{n=-\infty}^{\infty} q^{n(n+1)} z^n \sum_{k=0}^{\infty} \frac{q^{(k+1)^2}}{(q)_{2k+1}} z^{-k} ,$$

where

$$(q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n) .$$

Solution by the Proposer.

We shall make use of the Euler identity

$$\prod_{n=0}^{\infty} (1 - q^n z) = \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} z^n / (q)_n$$

and the Jacobi identity

$$\prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1} z)(1 - q^{2n-1} z^{-1}) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n .$$

Now we have

$$\sum_{n=0}^{\infty} \frac{q^{n^2}z^n}{q^n} \prod_{k=1}^{\infty} (1 - q^k) = \sum_{n=0}^{\infty} q^{n^2}z^n \prod_{k=1}^{\infty} (1 - q^{n+k})$$

$$= \sum_{n=-\infty}^{\infty} q^{n^2}z^n \prod_{k=1}^{\infty} (1 - q^{n+k})$$

$$= \sum_{n=-\infty}^{\infty} q^{n^2}z^n \sum_{k=0}^{\infty} (-1)^k q^{\frac{1}{2}k(k+1)+nk}/(q)_k$$

$$= \sum_{k=0}^{\infty} (-1)^k q^{\frac{1}{2}k(k+1)}/(q)_k \sum_{n=-\infty}^{\infty} q^{n^2}(q^k z)^n$$

$$= \sum_{n=0}^{\infty} (-1)^k q^{\frac{1}{2}k(k+1)}/(q)_k \cdot \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n+k-1}z)(1 + q^{2n-k-1}z^{-1})$$

$$= \prod_{n=1}^{\infty} (1 - q^{2n}) \cdot \sum_{k=0}^{\infty} \frac{q^{k(2k+1)}}{(q)_{2k}} \prod_{n=1}^{\infty} (1 + q^{2n+2k-1}z)(1 + q^{2n-2k-1}z)$$

$$- \prod_{n=1}^{\infty} (1 - q^{2n}) \cdot \sum_{k=0}^{\infty} \frac{q^{(k+1)(2k+1)}}{(q)_{2k+1}} \prod_{n=1}^{\infty} (1 + q^{2n+2k})(1 + q^{2n-2k-2}z^{-1})$$

$$= \sum_{n=-\infty}^{\infty} q^{n^2}z^n \sum_{k=0}^{\infty} \frac{q^{k(2k+1)}}{(q)_{2k}} \cdot \frac{(1 + q^{-2k+1}z^{-1})\cdots(1 + q^{-1}z^{-1})}{(1 + qz)\cdots(1 + q^{2k-1}z)}$$

$$- \sum_{n=-\infty}^{\infty} q^{n(n+1)}z^n \sum_{k=0}^{\infty} \frac{q^{(k+1)(2k+1)}}{(q)_{2k+1}} \cdot \frac{(1 + q^{-2k}z^{-1})\cdots(1 + q^{-2}z^{-1})}{(1 + q^2z)\cdots(1 + q^{2k}z)}$$

$$= \sum_{n=-\infty}^{\infty} q^{n^2}z^n \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q)_{2k}} z^{-k} - \sum_{n=-\infty}^{\infty} q^{n(n+1)}z^n \sum_{k=0}^{\infty} \frac{q^{(k+1)^2}}{(q)_{2k+1}} z^{-k} .$$

STAY TUNED TO THIS NETWORK

H-157 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada (corrected)

A set of polynomials $c_n(x)$, which appears in network theory is defined by

$$c_{n+1}(x) = (x + 2)c_n(x) - c_{n-1}(x) \qquad (n \geq 1)$$

with

$$c_0(x) = 1 \quad \text{and} \quad c_1(x) = (x + 2)/2 .$$

(a) Find a polynomial expression for $c_n(x)$.

(b) Show that

$$2c_n(x) = b_n(x) + b_{n-1}(x) = B_n(x) - B_{n-2}(x) ,$$

where $B_n(x)$ and $b_n(x)$ are the Morgan-Voyce polynomials as defined in the Fibonacci Quarterly, Vol. 5, No. 2, p. 167.

(c) Show that $2c_n^2(x) - c_{2n}(x) = 1$.

(d) If

$$Q = \begin{bmatrix} (x + 2) & -1 \\ 1 & 0 \end{bmatrix} ,$$

show that

$$\begin{bmatrix} c_n & -c_{n-1} \\ c_{n-1} & -c_{n-2} \end{bmatrix} = \frac{1}{2}(Q^n - Q^{n-2}) \quad \text{for} \quad (n \geq 2) .$$

Hence deduce that $c_{n+1}c_{n-1} - c_n^2 = x(x + 4)/4$.

Solution by A. G. Law, University of Saskatchewan, Regina, Saskatchewan, Canada.

Let $\{c_n(x)\}$ be the family of polynomials prescribed by the recurrence

$$(*) \qquad y_{n+1} = (x + 2)y_n - y_{n-1}, \qquad n \geq 1,$$

with $y_0 = 1$ and $y_1 = 1 + x/2$. It can be derived, with the aid of [1], that

$$c_n(x) = \frac{4^n(n!)^2}{(2n)!} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(1 + x/2), \qquad n \geq 1,$$

where $P_n^{(-1/2,-1/2)}$ is the n^{th}-degree Jacobi polynomial. Consequently [3], $c_n(x) = \cos n\theta$, where $\cos\theta = 1 + x/2$, for $n \geq 1$.

A half-angle formula gives immediately that $2c_n^2 - c_{2n} \equiv 1$, $n \geq 1$. Similarly, each relation

$$c_{n+1}(x)c_{n-1}(x) - c_n^2(x) = x(x + 4)/4$$

is also just a trigonometric identity.

The coupled recurrence

$$b_n = xB_{n-1} + b_{n-1}; \qquad B_n = (x + 1)B_{n-1} + b_{n-1} \qquad (n \geq 1),$$

where $b_0 \equiv B_0 \equiv 1$ shows that

$$b_{n+1} = (x + 2)b_n - b_{n-1}$$

for $n \geq 1$. Hence,

$$b_{n+1} = (x + 1)(b_n + b_{n-1}) - b_{n-2};$$

that is, $y_n = (b_n + b_{n-1})/2$ satisfies recurrence (*) and, so,

$$(b_n + b_{n-1})/2 \equiv c_n$$

for $n \geq 1$. Similarly, $2c_n \equiv B_n - B_{n-1}$ for $n \geq 1$.

Finally, since each $b_n(x)$ is a known sum (see [2]), $2c_n = b_n + b_{n-1}$ yields the explicit formula:

$$c_n(x) = x^n/2 + \sum_{k=0}^{n-1} \frac{n}{n-k}\binom{n+k-1}{n-k-1}x^k$$

for $n \geq 1$.

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2. J. C. Sjoherg, Problem H-69, Fibonacci Quarterly, Vol. 5 (1967), No. 2, pp. 164-165.
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Also solved by D. Zeitlin, D. V. Jaiswal, M. Yoder, and the Proposer.

IN THEIR PRIME

H-158 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

If $f_n(x)$ be the Fibonacci polynomial as defined in H-127, show that

(a) For integral values of x, $f_n(x)$ and $f_{n+1}(x)$ are prime to each other.

(b)
$$\left\{1 + \sum_1^n (1/f_{2n-1}F_{2n+1})\right\}\left\{1 - x^2 \sum_1^n (1/f_{2n}f_{2n+2})\right\} = 1\,.$$

Solution by the Proposer.

(a) It may easily be established by induction that

$$f_{n+1}(x)f_{n-1}(x) - f_n^2(x) = (-1)^n\,.$$

Hence, for integral values of x, $f_n(x)$ and $f_{n+1}(x)$ are prime to each other.

(b) It may also be established by induction that

(1) $$f_{n+1}(x)f_{n-2}(x) - f_n(x)f_{n-1}(x) + (-1)^n x = 0 .$$

Hence,

$$x \frac{1}{f_{2n+1}f_{2n-1}} = \frac{f_{2n+2}}{f_{2n+1}} - \frac{f_{2n}}{f_{2n-1}} .$$

Thus,

$$x \sum_1^n \frac{1}{f_{2n+1}f_{2n-1}} = \frac{f_{2n+2}}{f_{2n+1}} - \frac{f_2}{f_1} = \frac{f_{2n+2}}{f_{2n+1}} - x .$$

Or,

(2) $$1 + \sum_1^n \frac{1}{f_{2n+1}f_{2n-1}} = \frac{1}{x} \frac{f_{2n+2}}{f_{2n}} .$$

Also, from (1), we have

$$-x \frac{1}{f_{2n}f_{2n+2}} = \frac{f_{2n+1}}{f_{2n+2}} - \frac{f_{2n+1}}{f_{2n}} .$$

Hence,

$$-x \sum_1^n \frac{1}{f_{2n}f_{2n+2}} = \frac{f_{2n+1}}{f_{2n+2}} - \frac{f_3}{f_2}$$

$$= \frac{x f_{2n+2} + f_{2n+1}}{f_{2n+2}} - \frac{f_3}{f_2}$$

$$= \frac{f_{2n+1}}{f_{2n+2}} - \frac{x^2 + 1}{x} + x = \frac{f_{2n+1}}{f_{2n+2}} - \frac{1}{x} .$$

Thus,

(3) $$1 - x^2 \sum_1^n \frac{1}{f_{2n}f_{2n+2}} = x\,\frac{f_{2n+1}}{f_{2n+2}} .$$

Hence from (2) and (3), we have

$$\left\{1 + \sum_1^n \frac{1}{f_{2n+1}f_{2n-1}}\right\}\left\{1 - x^2 \sum_1^n \frac{1}{f_{2n}f_{2n+2}}\right\} = 1 .$$

Also solved by A. Shannon, M. Yoder, and D. V. Jaiswal.

HARMONY

H-159 Proposed by Clyde Bridger, Springfield College, Springfield, Illinois.

Let

$$D_k = \frac{c^k - d^k}{c - d}$$

and

$$E_k = c^k + d^k ,$$

where c and d are the roots of $z^2 = az + b$. Consider the four numbers e, f, x, y, where $e = c^k$ and $f = d^k$ are the roots of

$$z^2 - zE_k + (-b)^k = 0 ,$$

and y is the harmonic conjugate of x with respect to e and f. Find y when

$$x = \frac{D_{nk+k}}{D_{nk}} \qquad (k \neq 0) .$$

Solution by the Proposer.

The condition to be met is

$$\frac{x - e}{x - f} \cdot \frac{y - f}{y - e} = -1 .$$

(See page 69, R. M. Winger, Projective Geometry, Heath, 1923.) This leads directly to

$$2xy - E_k(x + y) + 2(-b)^k = 0 .$$

For the given value of x,

$$y = \frac{E_k D_{nk+k} - 2(-b)^k D_{nk}}{2 D_{nk+k} - E_k D_{nk}} .$$

It is easy to verify from the definitions of D_k and E_k that the numerator reduces to $E_{nk+k}D_k$ and that the denominator reduces to $E_{nk}D_k$. Hence,

$$y = \frac{E_{nk+k}}{E_{nk}} .$$

Note that when $a = b = 1$, and $k = 1$,

$$\frac{F_{n+1}}{F_n} \quad \text{and} \quad \frac{L_{n+1}}{L_n}$$

are harmonic conjugates with respect to the roots of $z^2 = z + 1$.

Also solved by M. Yoder.

DISCRIMINATING

H-160 Proposed by D. and E. Lehmer, University of California, Berkeley, California.

Find the roots and the discriminant of

$$x^3 - (-1)^k 3x - L_{3k} = 0 .$$

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Somewhat more generally, we may consider the equation

$$(*) \qquad x^3 - 3(\alpha\beta)^k x - (\alpha^{3k} + \beta^{3k}) = 0 ,$$

where α, β are arbitrary. This equation evidently reduces to

$$x^3 - 3(-1)^k x - L_{3k} = 0 ,$$

where α, β are the roots of

$$z^2 - z - 1 = 0 .$$

Let ω, ω^2 denote the complex cube roots of 1 and put

$$x_1 = \alpha^k + \beta^k, \qquad x_2 = \omega\alpha^k + \omega^2\beta^k, \qquad x_3 = \omega^2\alpha^k + \omega\beta^k .$$

Then it is easily verified that x_1, x_2, x_3 are the roots of (*).

By the familiar formula for the discriminant of a cubic, or directly by computing $(x_1 - x_2)^2(x_2 - x_3)^2$, we find that the discriminant is given by

$$D = -27(\alpha^{3k} - \beta^{3k})^2 .$$

For the special case

$$x^3 - 3(-1)^k x - L_{3k} = 0 ,$$

the roots are $x_1 = L_k$ and x_2, x_3, where

$$x_2 + x_3 = -L_k, \qquad x_2x_3 = L_{2k} - (-1)^k .$$

The discriminant reduces to

$$-135\,F_{3k}^2 .$$

Also solved by M. Yoder, D. Zeitlin, B. King, A. Shannon, and the Proposers.

BE NEGATIVE

H-162 Proposed by David A. Klarner, University of Alberta, Edmonton, Alberta, Canada.

Suppose $a_{ij} \geq 1$ for $i,j = 1, 2, \cdots$. Show there exists an $x \geq 1$ such that

$$(-1)^n \begin{vmatrix} a_{11} - x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - x^2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - x^n \end{vmatrix} \leq 0$$

for all n.

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania.

Let $D(n)$ be the determinant.

$$D(1) = (-1)^1 |a_{11} - x| = x - a_{11} \leq 0$$

if $x \leq a_{11}$. Since $x,\ a_{11} \geq 1$, any x satisfying $a_{11} \geq x \geq 1$ will do. Suppose $a_{11} = 1$; then $x = 1$ is the only answer for $n = 1$. The statement requires an x for all n. Can we reach a contradiction in the case $a_{11} = 1$? While

$$D(2) = -a_{12}\,a_{21} \leq -1 < 0 ,$$

$$D(3) = -\begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} - 1 & a_{23} \\ a_{31} & a_{32} & a_{33} - 1 \end{vmatrix} \doteq a_{21}a_{33} - a_{21} - a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31} - a_{13}a_{31}\,.$$

Each term here has the sign preceding it, as all factors are positive. Given a_{ij} with $i \neq j$, we can take a_{22} and/or a_{33} so large that the positive terms dominate, since these factors occur only in positive terms. Thus we reach a contradiction of the inequality for $n = 3$, $a_{11} = 1$.

[Continued from page 60.]

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