GREATEST COMMON DIVISORS IN ALTERED FIBONACCI SEQUENCES

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Let the Fibonacci and Lucas sequence be defined as usual:

$$F_{n+1} = F_n + F_{n-1}, \qquad L_{n+1} = L_n + L_{n-1}, \qquad n = 2, 3, \cdots,$$
$$F_1 = F_2 = L_1 = 1, \qquad L_2 = 3 .$$

It is well known that successive members of the Fibonacci sequence are relatively prime, but if we alter the sequence slightly by letting

$$G_n = F_n + (-1)^n, \qquad n = 1, 2, \cdots ,$$

then we have very different behavior, as can be seen in the following table:

n	1	2	3	4	5	6	7	8	9	10
G_n	0	2	1	4	4	9	12	22	33	56
(G_n, G_{n+1})		1		4		3		11		8

n	11	12	13	14	15	16	17	18	19
G_n	88	145	232	378	609	988	1596	2585	4180
(G_n, G_{n+1})		29		21		76		55	

Inspection of the table shows that the first, third, fifth, $\cdots$ entries in the (G_n, G_{n+1}) line are the second, fourth, sixth, $\cdots$ Fibonacci numbers, and the second, fourth, sixth, $\cdots$ entries are the third, fifth, seventh, $\cdots$ Lucas numbers. It is the purpose of this note to prove this and some related results which are corollaries of Theorem 1 below.

<u>Theorem 1:</u>

(1) $$F_{4n} + 1 = F_{2n-1}L_{2n+1}, \qquad F_{4n} - 1 = F_{2n+1}L_{2n-1},$$

(2) $$F_{4n+1} + 1 = F_{2n+1}L_{2n}, \qquad F_{4n+1} - 1 = F_{2n}L_{2n+1} ,$$

(3) $$F_{4n+2} + 1 = F_{2n+2}L_{2n} , \qquad F_{4n+2} - 1 = F_{2n}L_{2n+2} ,$$

(4) $$F_{4n+3} + 1 = F_{2n+1}L_{2n+2}, \qquad F_{4n+3} - 1 = F_{2n+2}L_{2n+1},$$

$n = 1, 2, \cdots$.

Proof. From [1, p. 59], we have

$$F_{n+p} + F_{n-p} = F_nL_p, \quad p \text{ even}, \tag{5}$$

$$F_{n+p} + F_{n-p} = F_pL_n, \quad p \text{ odd}, \tag{6}$$

$$F_{n+p} - F_{n-p} = F_pL_n, \quad p \text{ even}, \tag{7}$$

$$F_{n+p} - F_{n-p} = F_nL_p, \quad p \text{ odd}. \tag{8}$$

Using (6), we get

$$F_{4n} + 1 = F_{4n} + F_2 = F_{(2n+1)+(2n-1)} + F_{(2n+1)-(2n-1)}$$
$$= F_{2n-1}L_{2n+1} .$$

Using (5), we get

$$F_{4n+1} + 1 = F_{4n+1} + F_1 = F_{(2n+1)+2n} + F_{(2n+1)-2n}$$
$$= F_{2n+1}L_{2n} .$$

Similar applications of (5)-(8) give the remaining six identities in (1)-(4).

Although it is not known whether or not the Fibonacci sequence contains infinitely many primes, Theorem 1 shows that the sequences $\{F_n + 1\}$ and $\{F_n - 1\}$ contain only finitely many primes.

Corollary 1. $F_n + 1$ is composite for $n \geq 4$ and $F_n - 1$ is composite for $n \geq 7$.

Proof. From Theorem 1, $F_8 \pm 1$, $F_9 \pm 1$, $F_{10} \pm 1, \cdots$ are all composite because all of the factors on the right-hand sides of the equations in (1)-(4) are greater than one. Inspection of early values of F_n then completes the proof.

The property of greatest common divisors noted at the beginning of this note is proved in

Corollary 2.

$$(G_{4n}, G_{4n+1}) = L_{2n+1}, \qquad (G_{4n+2}, G_{4n+3}) = F_{2n+2},$$
$$(G_{4n+1}, G_{4n+3}) = L_{2n+1},$$

$n = 1, 2, \cdots$.

Proof. From Theorem 1, we have

$$(G_{4n}, G_{4n+1}) = (F_{4n} + 1,\ F_{4n+1} - 1) = (L_{2n+1}F_{2n-1}, L_{2n+1}F_{2n})$$
$$= L_{2n+1}(F_{2n-1}, F_{2n}) = L_{2n+1}\,.$$

The proofs of other equations are similar, the last one needing the fact that $(F_{2n}, F_{2n+2}) = 1$, $n = 1, 2, \cdots$.

Using Theorem 1 in a similar way, we can prove

Corollary 3. If $H_n = F_n - (-1)^n$, $n = 1, 2, \cdots$, then

$$(H_{4n}, H_{4n+1}) = F_{2n+1}, \qquad (H_{4n+2}, H_{4n+3}) = L_{2n+2}\,,$$
$$(H_{4n+1}, H_{4n+3}) = F_{2n+1}\,,$$

$n = 1, 2, \cdots$.

It would be natural to now consider the sequences $\{L_n + (-1)^n\}$ and $\{L_n - (-1)^n\}$, but different methods are needed.

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Note. The readers may wish to prove the additional ones listed below. Editor.

A. $(F_{4n+1} + 1,\ F_{4n+2} + 1) = L_{2n}$,

B. $(F_{4n+1} + 1,\ F_{4n+3} + 1) = F_{2n+1}$,

C. $(F_{4n+1} - 1,\ F_{4n+2} - 1) = F_{2n}$,

D. $(F_{4n+1} - 1,\ F_{4n+3} - 1) = L_{2n+1}$,

E. $(F_{4n-1} - 1,\ F_{4n+1} - 1) = F_{2n}$,

F. $(F_{4n-1} + 1,\ F_{4n+1} + 1) = L_{2n}$,

G. $(F_{4n+3} + 1,\ F_{4n} - 1) = F_{2n+1}$,

H. $(F_{4n+3} + 1,\ F_{4n+2} - 1) = F_{2n}$,

I. $(F_{4n+4} - 1,\ F_{4n+3} - 1) = L_{2n+1}$.

REFERENCE

1. Verner E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton Mifflin, Boston, 1969.