BINET FORMS BY LAPLACE TRANSFORM

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In past articles of the Fibonacci Quarterly, several methods have been suggested for solutions to n^{th}-order difference equations.

In a series of articles entitled "Linear Recursive Relations," J. A. Jeske attacks and solves this problem by use of generating functions [1, p. 69], [2, p. 35], [3, p. 197].

In another series of articles also entitled, "Linear Recursive Relations," Brother Alfred Brousseau, one of the founders of the Fibonacci Quarterly, outlines a method of finding Binet forms using matrices [4, p. 99], [5, p. 194], [6, p. 295], [7, p. 533].

What I propose to do here is to find a general solution to the linear homogenous difference equation with distinct roots to the characteristic. The method of solution will be Laplace Transform.

Unfortunately, the Laplace Transform does not deal with discrete functions. So, to make the problem applicable, define the continuous function $y(t)$ such that $y(t) = a_n \quad n \le t < n+1 \quad n = 0, 1, 2, \cdots$, where a_n, $n \in Z$, is the sequence of the difference equation. This changes the discrete sequence to a continuous and integrable function.

The following is the Laplace Transform pair:

$$Y(s) = \int_0^\infty e^{-st} y(t)\, dt$$

$$y(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} Y(s)\, ds \quad .$$

The inversion formula is messy. It is a contour integral, and requires a knowledge of complex variables. In our case, we will "recognize" the resultant inverse. The following Lemma illustrates the integration of our step function $y(t)$, and will be used in a subsequent theorem.

Lemma 1. If $y(t) = a_n$, $n \le t < n+1$, $n = 0, 1, 2, \cdots$, then

$$L\{y(t+j)\} = e^{sj}Y(s) - \left(\frac{1-e^{-s}}{s}\right)\sum_{n=0}^{j-1} a_n e^{s(j-n)} .$$

Proof. By definition,

$$L\{y(t+j)\} = \int_0^\infty y(\beta + j)e^{-s\beta}d\beta .$$

Let $\beta + j \to t$. Then

$$L\{y(t+j)\} = \int_j^\infty y(t)e^{-s(t-j)}dt$$

$$= e^{sj}\int_0^\infty y(t)e^{-st}\,dt.$$

$$= e^{sj}\int_0^\infty y(t)e^{-st}dt - e^{sj}\int_0^j y(t)e^{-st}dt$$

$$= e^{sj}Y(s) - e^{sj}\sum_{n=0}^{j-1} a_n \int_n^{n+1} e^{-st}dt ,$$

since $y(t) = a_n$ $n \le t < n+1$,

$$= e^{sj}Y(s) - e^{sj}\sum_{n=0}^{j-1} a_n \left(e^{-sn}\left(\frac{1-e^{-s}}{s}\right)\right)$$

$$= e^{sj}Y(s) - \left(\frac{1-e^{-s}}{s}\right)\sum_{n=0}^{j-1} a_n e^{s(j-n)}$$

The next Lemma will provide the inverse that we will later "recognize."

Lemma 2. If $y(t) = \alpha^n$, $n \le t < n+1$, $n = 0, 1, 2, \cdots$, where α is a constant, then

$$Y(s) = \left(\frac{1 - e^{-s}}{s}\right)\left(\frac{1}{1 - \alpha e^{-s}}\right)$$

Proof. By definition,

$$\begin{aligned} Y(s) &= \int_0^\infty y(t)e^{-st}dt \\ &= \sum_{n=0}^\infty \int_n^{n+1} \alpha^n e^{-st}dt \\ &= \sum_{n=0}^\infty \alpha^n e^{-sn}\left(\frac{1 - e^{-s}}{s}\right) \\ &= \left(\frac{1 - e^{-s}}{s}\right)\sum_{n=0}^\infty \left(\alpha e^{-s}\right)^n \\ &= \left(\frac{1 - e^{-s}}{s}\right)\left(\frac{1}{1 - \alpha e^{-s}}\right) \end{aligned}$$

The third and last Lemma is a very slight modification to the Partial Fractions Theorem to fit our particular needs. Here $Q(x)$ has distinct roots α_i.

Lemma 3. Let $Q(x)$ be a polynomial, degree N. Let $P(x)$ be a polynomial, degree $\le N$. Then if

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^N \frac{\gamma_i}{(1 - \alpha_i x^{-1})}, \quad \gamma_i = \frac{P(\alpha_i)}{\alpha_i Q'(\alpha_i)} .$$

Proof. Let

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^N \frac{\gamma_i}{(1 - \alpha_i x^{-1})}$$

Then

$$P(x) = \sum_{i=1}^{N} \frac{\gamma_i Q(x)}{(1 - \alpha_i x^{-1})}$$

$$P(x) = \sum_{i=1}^{N} \frac{x\gamma_i Q(x)}{x - \alpha_i}$$

$$\lim_{x \to \alpha_j} P(x) = \sum_{i=1}^{N} \left[\left[\lim_{x \to \alpha_j} x \quad i\right]\left[\lim_{x \to \alpha_j} \frac{Q(x)}{(x - \alpha_i)}\right]\right]$$

The limit on the right is $Q'(\alpha_j)$ when $i = j$ and 0 otherwise. Therefore,

$$P(\alpha_j) = \alpha_j \gamma_j Q'(\alpha_j)$$

$$\Rightarrow \gamma_j = \frac{P(\alpha_j)}{\alpha_j Q'(\alpha_j)} \quad .$$

We now have sufficient information to solve the problem. First, we find the transform of the difference equation producing $\{a_n | n = Z^+\}$.

Theorem 1. If $y(t) = a_n$, $n \le t < n + 1$, $n = 0, 1, 2, \cdots$, and

$$\sum_{j=0}^{N} A_j y(t + j) = 0:$$

A_j are coefficients: N is the degree, then the transform

$$Y(s) = \left(\frac{1 - e^{-s}}{s}\right) \frac{\sum_{i=1}^{N} A_j \sum_{n=0}^{j-1} a_n e^{s(j-n)}}{\sum_{j=0}^{N} A_j e^{sj}} \quad .$$

Proof.

$$L\left\{\sum_{j=0}^{N} A_j y(t + j)\right\} = \sum_{j=0}^{N} A_j L\{y(t + j)\} = 0$$

$$A_0 Y(s) + \sum_{j=1}^{N} A_j L\{y(t + j)\} = 0$$

From Lemma 1,

$$A_0 Y(s) + \sum_{j=1}^{N} A_j \left[e^{sj} Y(s) - \sum_{n=0}^{j-1} a_n e^{s(j-n)} \left(\frac{1 - e^{-s}}{s}\right)\right] = 0$$

$$A_0 Y(s) + \sum_{j=1}^{N} A_j e^{sj} Y(s) = \sum_{j=1}^{N} A_j \sum_{n=0}^{j-1} a_n e^{s(j-n)} \left(\frac{1 - e^{-s}}{s}\right)$$

Y(s)

$$Y(s) = \left(\frac{1 - e^{-s}}{s}\right) \frac{\sum_{j=1}^{N} A_j \sum_{n=0}^{j-1} a_n e^{s(j-n)}}{\sum_{j=0}^{N} A_j e^{sj}}$$

The transform is actually a quotient of polynomials in e^s. The following is a corollary based on the previous theorem and Lemma 3. We get

Corollary. If $y(t) = a_n$, $n \le t < n + 1$, $n = 0, 1, 2, \cdots$, and

$$\sum_{j=0}^{N} A_j y(t + j) = 0$$

and the roots of

$$\sum_{j=0}^{N} A_j x^j$$

distinct (α_i), then

$$Y(s) = \left(\frac{1 - e^{-s}}{s}\right) \sum_{i=1}^{N} \frac{\gamma_i}{(1 - \alpha_i e^{-s})} \quad ,$$

where

$$\gamma_i = \frac{\sum_{j=1}^{N} A_j \sum_{n=0}^{j-1} a_n \alpha_i^{j-n}}{\sum_{j=1}^{N} j A_j \alpha_i^j} \quad .$$

Proof. Let $x = e^s$. Then

$$P(x) = \sum_{j=1}^{N} A_j \sum_{n=0}^{j-1} a_n x^{j-n}$$

$$Q(x) = \sum_{j=0}^{N} A_j x^j$$

$$Q'(x) = \sum_{j=1}^{N} j A_j x^{j-1} \quad .$$

Then if the roots of $Q(x)$, α_i, are distinct

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{N} \frac{\gamma_i}{1 - \alpha_i e^{-s}} \quad ,$$

where

$$\gamma_i = \frac{P(\alpha_i)}{\alpha_i Q'(\alpha_i)} = \frac{\sum_{j=1}^{N} A_j \sum_{n=0}^{j-1} a_n \alpha_i^{j-n}}{\alpha_i \sum_{j=1}^{N} j A_j \alpha_i^{j-1}}$$

Therefore,

$$Y(s) = \left(\frac{1 - e^{-s}}{s}\right)\sum_{i=1}^{N} \frac{\gamma_i}{(1 - \alpha_i e^{-s})} ,$$

where

$$\gamma_i = \frac{\sum_{j=1}^{j} A_j \sum_{n=0}^{j-1} a_n a_i^{j-n}}{\sum_{j=1}^{N} jA_j\alpha_i^j} .$$

The Corollary gives a very nice little package to unravel. Finding the inverse is a direct result of Lemma 2.

Theorem 2. If $y(t) = a_n$, $n \le t < n + 1$, $n = 0, 1, 2, \cdots$, and

$$\sum_{j=0}^{N} A_j y(t + j) = 0 ,$$

then

$$y(t) = \sum_{i=1}^{N} \gamma_i \alpha_i^n \qquad n = 0, 1, 2, \cdots ,$$

where

$$\gamma_i = \frac{\sum_{j=1}^{N} A_j \sum_{n=0}^{j-1} a_n \alpha_i^{j-n}}{\sum_{j=1}^{N} jA_j\alpha_i^j} .$$

The proof is implicit from the Corollary and Lemma 2. Consider the following problem of Pell:

$$P_{n+2} = 2P_{n+1} + P_n \qquad P_0 = 0, \quad P_1 = 1$$

Translating this into our terminology yields:

$$y(t + 2) - 2y(t + 1) - y(t) = 0 \qquad n \le t < n + 1 \qquad n = 0, 1, 2, \cdots$$

$$A_0 = -1 \qquad a_0 = 0$$
$$A_1 = -2 \qquad a_1 = 1$$
$$A_2 = 1$$

Since $\alpha^2 - 2\alpha - 1 = 0$, $\alpha = 1 \pm \sqrt{2}$. Let

$$\alpha_1 = 1 + \sqrt{2}$$
$$\alpha_2 = 1 - \sqrt{2} \quad .$$

Now from Theorem 2:

$$y(t) = \gamma_1\alpha_1^n + \gamma_2\alpha_2^n$$

$$\gamma_i = \frac{A_1 a_0 \alpha_i + A_2(a_0\alpha_i^2 + a_1\alpha_i)}{A_1\alpha_i + 2A_2\alpha_i^2} \quad .$$

After reducing with $a_0 = 0$,

$$\gamma_i = \frac{A_2 a_1}{2A_2\alpha_i + A_1} \qquad \text{or} \qquad \gamma_i = \frac{1}{2\alpha_i - 2}$$

Since $\alpha_1 = 1 + \sqrt{2}$,

$$\gamma_2 = \frac{1}{2(1 + \sqrt{2}) - 2} = -\frac{1}{2\sqrt{2}}$$

Since $\alpha_2 = 1 - \sqrt{2}$,

$$\gamma_2 = \frac{1}{2(1 - \sqrt{2}) - 2} = -\frac{1}{2\sqrt{2}}$$

Therefore,

$$y(t) = a_n = \frac{\alpha_1^n - \alpha_2^n}{2\sqrt{2}} ,$$

which of course we recognize is the Binet form for the Pell sequence. In fact, similarly we can find Binet forms for Fibonacci, Lucas, or any other Homogenous Linear Difference Equations where roots to $\Sigma_i A_i x^i$, the characteristic, are distinct.

One more logical extension of Fibonacci sequence is the Tribonacci. This problem is the Fibonacci equation extended to the next degree.

$$T_{n+2} = T_{n+2} + T_{n+1} + T_n .$$

In this instance, the most difficult part lies in solving the characteristic equation,

$$m^3 - m^2 - m - 1 = 0 ,$$

for its roots using Cardan formulae. This involves a little algebra and a little time. The procedure yields the roots,

$$\alpha_1 = \frac{1}{3} + \left(\frac{19}{27} + \frac{1}{3}\left(\frac{11}{3}\right)^{1/2}\right)^{1/3} + \left(\frac{19}{27} - \frac{1}{3}\left(\frac{11}{3}\right)^{1/2}\right)^{1/3}$$

$$\alpha_2 = \frac{1}{3} - \frac{1}{2}\left(\frac{19}{27} + \frac{1}{3}\left(\frac{11}{3}\right)^{1/2}\right)^{1/3} - \frac{1}{2}\left(\frac{19}{27} - \frac{1}{3}\left(\frac{11}{3}\right)^{1/2}\right)^{1/3}$$
$$+ \frac{i\sqrt{3}}{2}\left[\left(\frac{19}{27} + \frac{1}{3}\left(\frac{11}{3}\right)^{1/2}\right)^{1/3} - \left(\frac{19}{27} - \frac{1}{3}\left(\frac{11}{3}\right)^{1/2}\right)^{1/3}\right]$$

$$\alpha_3 = \bar{\alpha}_2$$

Now for those of us more furtunate fellows, we can simplify some of this by means of a computer, which yields:

$$\alpha_1 = 1.84$$
$$\alpha_2 = -0.42 + 0.61i$$
$$\alpha_3 = -0.42 - 0.61i$$

From Theorem 2,

$$A_3 = 1 \qquad a_0 = 1$$
$$A_2 = A_1 = A_0 = -1 \qquad a_1 = 0$$
$$a_2 = 0$$

$$\gamma_i = \frac{A_1 a_0 \alpha_i + A_2(a_0\alpha_i^2 + a_i\alpha_i) + A_3(a_0\alpha_i^3 + a_1\alpha_i^2 + a_2\alpha_i)}{A_1\alpha_i + 2A_2\alpha_i^2 + 3A_3\alpha_i^3}$$

Reduced, $(\alpha^3 = \alpha^2 + \alpha + 1)$

$$\gamma_i = \frac{1}{\alpha_i^2 + 2\alpha_i + 3} ,$$

Therefore,

$$y(t) = a_n = T_n = \left(\frac{1}{\alpha_1^2 + 2\alpha_1 + 3}\right)\alpha_1^n + \left(\frac{1}{\alpha_2^2 + 2\alpha_2 + 3}\right)\alpha_2^n + \left(\frac{1}{\alpha_3^2 + 2\alpha_3 + 3}\right)\alpha_3^n .$$

Now, you, too, can find your own Binet forms.

FOOD FOR THOUGHT

Brother Alfred Brousseau says, for $N = 2$,

$$\gamma_1 = \frac{a_0\alpha_2 - a_1}{\alpha_2 - \alpha_1} \qquad \gamma_2 = \frac{a_0\alpha_1 - a_1}{\alpha_1 - \alpha_2} ,$$

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