A SYMMETRIC SUBSTITUTE FOR STIRLING NUMBERS

A. P. HILLMAN, P. L. MANA, and C. T. McABEE
University of New Mexico, Albuquerque, New Mexico

Let

$$\binom{x}{0} = 1$$

and

$$\binom{x}{r} = \frac{x(x-1)(x-2)\cdots(x-r+1)}{1 \cdot 2 \cdot 3 \cdots r}$$

for all complex numbers x and all positive integers r. It is well known that

$$(1) \qquad \binom{0}{r} + \binom{1}{r} + \binom{2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1},$$

and that for every non-negative integer d there exist integers $s_{d0}, s_{d1}, \cdots, s_{dd}$ such that

$$(2) \qquad x^d = s_{d0}\binom{x}{0} + s_{d1}\binom{x}{1} + s_{d2}\binom{x}{2} + \cdots + s_{dd}\binom{x}{d}$$

holds for all x. (The s_{dj} are related to the Stirling numbers of the second kind.) Using (1) and (2), one obtains the summation formulas

$$(3) \qquad 0^d + 1^d + 2^d + \cdots + n^d = s_{d0}\binom{n+1}{1} + s_{d1}\binom{n+1}{2} + \cdots + s_{dd}\binom{n+1}{d+1}.$$

This paper presents alternates for (2) and (3) in which the s_{dj} are replaced by coefficients having symmetry properties and other advantages. Part of the work generalizes with the help of Dov Jarden's results from the $\binom{n}{r}$ to generalized binomial coefficients.

Using the well known

$$\binom{x}{r} + \binom{x}{r+1} = \binom{x+1}{r+1},$$

one easily proves

$$(4)\qquad \binom{x}{r-s} = \binom{s}{0}\binom{x+s}{r} - \binom{s}{1}\binom{x+s-1}{r} + \cdots + (-1)^s\binom{s}{s}\binom{x}{r}$$

by mathematical induction. Then (2) and (4) imply that for every non-negative integer d there exist integers a_{dj} such that

$$(5)\qquad x^d = s_{d0}\binom{x}{d} + a_{d1}\binom{x+1}{d} + \cdots + a_{dd}\binom{x+d}{d}.$$

From (1) and (5), one now obtains

$$(6)\qquad 0^d + 1^d + \cdots + n^d = a_{d0}\binom{x+1}{d+1} + a_{d1}\binom{x+2}{d+1} + \cdots + a_{dd}\binom{x+d+1}{d+1}.$$

For example,

$$x^2 = \binom{x}{2} + \binom{x+1}{2}, \qquad x^3 = \binom{x}{3} + 4\binom{x+1}{3} + \binom{x+2}{3},$$

$$x^4 = \binom{x}{4} + 11\binom{x+1}{4} + 11\binom{x+2}{4} + \binom{x+3}{4},$$

$$x^5 = \binom{x}{5} + 26\binom{x+1}{5} + 66\binom{x+2}{5} + 26\binom{x+3}{5} + \binom{x+4}{5},$$

$$0^2 + 1^2 + 2^2 + \cdots + n^2 = \binom{n+1}{3} + \binom{n+2}{3}.$$

and

$$\sum_{k=0}^{n} k^3 = \binom{n+1}{4} + 4\binom{n+2}{4} + \binom{n+3}{4}.$$

The listed cases of (5) suggest that the following may be true:

$$a_{dd} = 0 \tag{7}$$

$$a_{d0} = 1 = a_{d,d-1} \tag{8}$$

$$a_{dj} = a_{d,d-1-j} \tag{9}$$

$$a_{d0} + a_{d1} + \cdots + a_{d,d-1} = 1 \cdot 2 \cdot 3 \cdots d = d! \quad . \tag{10}$$

Successively letting x be $0, -1, 1, -2, 2, \cdots$ in (5) establishes (7), (8), and (with the help of mathematical induction) the symmetry formula (9). These substitutions also prove that the a_{dj} are unique. One obtains (10) from

$$\frac{1}{d+1} = \int_0^1 x^d dx = \lim_{n\to\infty} \left(\left[\left(\frac{1}{n}\right)^d + \left(\frac{2}{n}\right)^d + \cdots + \left(\frac{n}{n}\right)^d \right] \Big/ n \right)$$

$$= \lim_{n\to\infty} [(1^d + 2^d + \cdots + n^d)/n^{d+1}]$$

$$= [a_{d0} + a_{d1} + \cdots + a_{d,d-1}]/(d+1)! \quad .$$

A recursion formula for the a_{dj} is derived as follows:

$$\sum_{j=0}^{d} a_{d+1,j} \binom{x+j}{d+1} = x^{d+1}$$

$$= x \sum_{j=0}^{d-1} a_{dj} \binom{x+j}{d}$$

$$= \sum_{j=0}^{d-1} a_{dj} \, [(x-d+j) + (d-j)] \binom{x+j}{d}$$

$$= \sum_{j=0}^{d-1} a_{dj}\left[(d+1)\binom{x+j}{d+1} + (d-j)\binom{x+j}{d}\right]$$

$$= \sum_{j=0}^{d-1} a_{dj}\left[(d+1)\binom{x+j}{d+1} + (d-j)\binom{x+j+1}{d+1} - (d-j)\binom{x+j}{d+1}\right]$$

$$= \sum_{j=0}^{d-1} a_{dj}(j+1)\binom{x+j}{d+1} + \sum_{j=0}^{d-1} a_{dj}(d-j)\binom{x+j+1}{d+1}$$

$$= \sum_{j=0}^{d-1} a_{dj}(j+1)\binom{x+j}{d+1} + \sum_{k=1}^{d} a_{d,k-1}(d-k+1)\binom{x+k}{d+1}$$

$$= a_{d0}\binom{x}{d+1} + \sum_{j=1}^{d-1}\left[(j+1)a_{dj} + (d-j+1)a_{d,j-1}\right]\binom{x+j}{d+1}$$

$$+ a_{d,d-1}\binom{x+d}{d+1} .$$

This and uniqueness of the a_{dj} imply that for $j = 1, 2, \cdots, d-1$ one has

$$a_{d+1,j} = (j+1)a_{dj} + (d-j+1)a_{d,j-1} . \tag{11}$$

Using $a_{d0} = 1$ and (11) gives us $a_{d+1,1} = 2a_{d1} + d$. Let E be the operator on functions of d such that $Ey_d = y_{d+1}$. Then $(E-2)a_{d1} = d$ and

$$(E^2 - 2E + 1)(E - 2)a_{d1} = (d+2) - 2(d+1) + d = 0 .$$

It follows from the theory of linear homogeneous difference equations with constant coefficients that there are constants e_0, e_1, and e_2 such that

$$a_{d1} = e_0 + e_1 d + e_2 \cdot 2^d \quad \text{for} \quad d = 1, 2, 3, \cdots .$$

Using the known values of a_{11}, a_{21}, and a_{31} one solves for e_0, e_1, and e_2 and thus shows that $a_{d1} = 2^d - d - 1$. Similarly, one sees that

$$(E - 1)^3(E - 2)^2(E - 3)a_{d2} = 0$$

and hence that there are constants f_i such that

$$a_{d2} = (f_0 + f_1 d + f_2 d^2) + (f_3 + f_4 d)2^d + f_5 \cdot 3^d .$$

Determining the f_i, one finds that

$$a_{d2} = 3^d - \binom{d+1}{1} 2^d + \binom{d+1}{2} .$$

Now (or after additional cases) one conjectures that

$$a_{dj} = \sum_{k=0}^{j} (-1)^k (j + 1 - k)^d \binom{d+1}{k} . \tag{12}$$

Because of the symmetry formula (9), we know that (12) is equivalent to

$$a_{dj} = \sum_{k=0}^{d-j-1} (-1)^k (d - j - k)^d \binom{d+1}{k} . \tag{13}$$

Substituting (13) into (5) gives us

$$x^d = \sum_{j=0}^{d-1} \left\{ \left[\sum_{k=0}^{d-j-1} (-1)^k (d - j - k)^d \binom{d+1}{k} \right] \binom{x+j}{d} \right\} . \tag{14}$$

Since the a_{dj} that satisfy (5) are unique, one can prove (13) and (12) by showing that (14) is an identity in x. Since both sides of (14) are polynomials in x of degree d, it suffices to verify (14) for the $d+1$ values $x = 0, 1, \cdots, d$. For such an x, (14) becomes

$$(15) \qquad x^d = x^d + \sum_{r=1}^{x-1} \left\{ r^d \sum_{j=0}^{x-r} (-1)^j \binom{d+1}{j} \binom{x+d-r-j}{d} \right\} .$$

Since $\binom{h}{d} = 0$ for $h = 0, 1, \cdots, d-1$, one has

$$(16) \qquad \sum_{j=0}^{x-r} (-1)^j \binom{d+1}{j} \binom{x+d-r-j}{d} = \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} \binom{x+d-r-j}{d} .$$

The right side sum in (16) is zero since it is a $(d+1)^{st}$ difference of a polynomial of degree d. Hence (15) becomes the tautology $x^d = x^d$. This establishes (13) and (12).

We next apply some of the above material to convolution formulas. It is well known (and easily shown by Maclaurin's expansion or Newton's binomial expansion) that

$$(17) \qquad (1-x)^{-d-1} = \binom{d}{d} + \binom{d+1}{d} x + \binom{d+2}{d} x^2 + \cdots \quad \text{for} \quad -1 < x < 1 .$$

Using (5) and (17), we obtain

$$(18) \qquad (a_{d,d-1} + a_{d,d-2}x + \cdots + a_{d,0}x^{d-1})(1-x)^{-d-1} = 1^d + 2^d x + 3^d x^2 + \cdots , \quad |x| < 1 .$$

Now let

$$(19) \qquad p(d,x) = a_{d,0} + a_{d,1}x + \cdots + a_{d,d-1}x^{d-1} = a_{d,d-1} + a_{d,d-2} + \cdots + a_{d,0}x^{d-1} .$$

Then (18) can be rewritten as

$$p(d,x)(1-x)^{-d-1} = \sum_{j=0}^{\infty} (j+1)^d x^j, \qquad |x| < 1. \tag{20}$$

Also let

$$p(d,x)p(e,x) = q(d,e,x) = c_{d,e,0} + c_{d,e,1}x + \cdots + c_{d,e,d+e-2}x^{d+e-2}. \tag{21}$$

Then

$$\sum_{k=0}^{n} k^d (n-k)^e \tag{22}$$

is the coefficient of x^n in the Maclaurin expansion of

$$q(d,e,x)(1-x)^{-d-e-2},$$

i.e., (22) is equal to

$$\sum_{j=0}^{d+e-2} c_{d,e,j} \binom{n+1+j}{d+e+1}. \tag{23}$$

For example, since $p(3,x) = 1 + 4x + x^2$, and $p(2,x) = 1 + x$, we have

$$q(3,2,x) = (1 + 4x + x^2)(1 + x) = 1 + 5x + 5x^2 + x^3,$$

and it follows from the equality of (22) and (23) that

$$\sum_{k=0}^{n} k^3 (n-k)^2 = \binom{n+1}{6} + 5\binom{n+2}{6} + 5\binom{n+3}{6} + \binom{n+4}{6}.$$

We note that the recursion formula (11) for the a_{dj} can also be derived from (20) using

$$(24) \qquad d[xp(d,x)(1-x)^{-d-1}]/dx = p(d+1,x)(1-x)^{-d-2} .$$

Next we turn to generalizations of (5) and (12) in which the sequence $0, 1, 2, 3, \cdots$ is replaced by any sequence $U_0, U_1, U_2, U_3, \cdots$ satisfying

$$(25) \quad U_0 = 0, \; U_1 = 1, \; U_{n+2} = gU_{n+1} - hU_n \quad \text{for} \quad n = 0, 1, 2, \cdots , \quad \text{and} \; h^2 = 1 .$$

The following table indicates some of the well-known sequences that are included for special values of g and h:

g	h	Sequence
2	1	Natural Numbers: $U_n = n$
1	-1	Fibonacci Numbers: $U_n = F_n$
2	-1	Pell Numbers: $U_n = P_n$
L_k	$(-1)^k$	$U_n = F_{kn}/F_k$

A key formula for the generalized sequence U_n is the addition formula

$$U_2U_{m+n+2} = U_{m+2}U_{n+2} - U_mU_n ,$$

which is established by double induction using (25) and verification for the four cases in which (m,n) is $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$.

We now assume that (g,h) is not $(1,1)$ in (25); then (25) is ordinary in the sense of Torretto-Fuchs (see [1]) and so $U_n \neq 0$ for $n > 0$. Then we use the Torretto-Fuchs notation

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} n \\ r \end{bmatrix} = \frac{U_mU_{n-1} \cdots U_{n-r+1}}{U_1U_2 \cdots U_r} \quad \text{for} \quad r = 1, 2, \cdots$$

for Dov Jarden's generalized binomials. Jarden showed 2 that

$$\sum_{j=0}^{d+1} (-1)^j h^{j(j-1)/2} \begin{bmatrix} d+1 \\ j \end{bmatrix} Z_{n-j} = 0 \,, \tag{26}$$

if Z_n is the term-by-term product of the n^{th} terms of d sequences each of which satisfies the recursion formula (25). The sequence $Z_n = \begin{bmatrix} n \\ d \end{bmatrix}$ is such a product, hence

$$\sum_{j=0}^{d+1} (-1)^j h^{j(j-1)/2} \begin{bmatrix} d+1 \\ j \end{bmatrix} \begin{bmatrix} n-j \\ d \end{bmatrix} = 0 \,. \tag{27}$$

We are now in a position to give the following generalizations of (5) and (12):

$$U_n^d = B_{d0} \begin{bmatrix} n+d-1 \\ d \end{bmatrix} + B_{d1} \begin{bmatrix} n+d-2 \\ d \end{bmatrix} + \cdots + B_{d,d-1} \begin{bmatrix} n \\ d \end{bmatrix} \,, \tag{28}$$

where

$$B_{dj} = \sum_{k=0}^{d-j-1} (-1)^k h^{k(k-1)/2} U_{j+1-k}^d \begin{bmatrix} d+1 \\ k \end{bmatrix} \tag{29}$$

Formula (29) is established in the same fashion as for formula (12), with the vanishing of the sums of (16) replaced by (27).

We do not generalize the summation formula (6) since we are not able to give a generalization of formula (1). However, we do present the following summation formulas involving the generalized sequence U_n :

$$U_2 + U_4 + \cdots + U_{2n} = (U_{n+1}^2 + U_n^2 - U_1^2)/U_2 \tag{30}$$

$$U_1U_3 + U_2U_6 + \cdots + U_nU_{3n} = U_nU_{n+1}U_{2n+1}/U_2 \tag{31}$$

(32) $$U_1^2U_2 + U_2^2U_4 + \cdots + U_n^2U_{2n} = U_n^2U_{n+1}^2/U_2 .$$

These formulas are easily probed by mathematical induction using the following special cases of the above addition formula:

(33) $$U_{n+2}^2 - U_n^2 = U_2U_{2n+2}$$

(34) $$U_{n+2}U_{2n+3} - U_nU_{2n+1} = U_2U_{3n+3} .$$

The special case of (31) in which $U_n = F_n$ is Recke's problem [3] which brought to mind the well-known formula

(35) $$1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6 .$$

These two special cases inspired the generalization (31). Then (32) was obtained as a generalization of the well-known

(36) $$1^3 + 2^3 + 3^3 + \cdots + n^3 = n^2(n + 1)^2/2 .$$

The proofs of (31) and (32) produced (33) as a byproduct; then (30) follows readily using the telescoping sum

$$\sum_{k=1}^{n} [U_2U_{2k}] = \sum_{k=0}^{n} [U_{k+1}^2 - U_{k-1}^2] = U_{n+1}^2 + U_n^2 - U_1^2 .$$

Some special cases of (28) and a special case of (33) above were proposed by one of the authors [4].

Formulas (5), (11), and (13) go back to J. Worpitzky and G. Frobenius (see [5] and [6]). These have been generalized in a different manner from our formulas (28) and (29) by L. Carlitz [7].

[Continued on page 73.]