TWENTY-FOUR MASTER IDENTITIES

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1. INTRODUCTION

The area of Fibonacci research is expanding and generalized, and a large number of known identities have been listed in many articles in these pages and in the booklet [1]. Many new results and old will be summarized in the forthcoming Concordance, edited by George Ledin, Jr., to appear in 1971. Here, we generalize the results of John Halton [2]. Leonard in his thesis [3] also expanded upon this in several directions. David Zeitlin has promised an all-encompassing paper to follow upon this generalization theme.

2. THE HILBERT TENTH PROBLEM

In [4] Matijasevic proves Lemma 17: $F_m^2 | F_{mr}$ iff $F_m | r$. At the end of the English translation, the translators suggest a sequence of lemmas leading to a simplified derivation. We now prove it in an even simpler way.

Let

$$\alpha = \frac{1 + \sqrt{5}}{2}, \qquad \text{and} \qquad \beta = \frac{1 - \sqrt{5}}{2},$$

then

$$\alpha^m = \alpha F_m + F_{m-1} \qquad \text{and} \qquad \beta^m = \beta F_m + F_{m-1}.$$

Recall

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta};$$

then

$$F_{mr} = \frac{\alpha^{mr} - \beta^{mr}}{\alpha - \beta} = \sum_{k=0}^{r} \binom{r}{k} F_m^k F_{m-1}^{r-k} \frac{(\alpha^k - \beta^k)}{\alpha - \beta}$$

$$= \sum_{k=0}^{r} \binom{r}{k} F_m^k F_{m-1}^{r-k} F_k \quad .$$

Next, $F_0 = 0$, and F_m^2 divides all terms for $k \geq 2$. Thus,

$$F_{mr} \equiv \binom{r}{1} F_m F_{m-1}^{r-1} F_1 \equiv rF_m F_{m-1}^{r-1} \pmod{F_m^2}$$

Since $(F_m, F_{m-1}) = 1$, then the result follows easily. A similar result could have been derived from

$$\alpha^m = F_{m+1} - \beta F_m \quad \text{and} \quad \beta^m = F_{m+1} - \alpha F_m \ .$$

3. THE DERIVATIONS

Let $\alpha^k = AF_{k+t} + BF_k$. Then,

$$\sqrt{5}\alpha^k = A(\alpha^{k+t} - \beta^{k+t}) + B(\alpha^k - \beta^k)$$
$$= \alpha^k(A\alpha^t + B) - \beta^k(A\beta^t + B)$$

Therefore,

$$\sqrt{5} = A\alpha^t + B$$
$$0 = A\beta^t + B \ ,$$

and

$$A = \sqrt{5}/(\alpha^t - \beta^t) = 1/F_t; \qquad B = -\beta^t A = -\beta^t/F_t \ ,$$

and thus

(1) $$F_{k+t} = \alpha^k F_t + \beta^t F_k \ .$$

Since k and t are arbitrary integers, we may interchange them:

(2) $$F_{k+t} = \beta^k F_t + \alpha^t F_k \; .$$

Equation (1) yields

(3) $$\alpha^j F_{k+t}^n = \alpha^j \sum_{i=0}^{n} \binom{n}{i} F_t^{n-i} F_k^i \alpha^{k(n-i)} \beta^{ti} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{ti} F_t^{n-i} F_k^i \alpha^{k(n-i)-ti+j} \; ,$$

and, in a similar manner, Eq. (2) gives us:

(4) $$\beta^j F_{k+t}^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{ti} F_t^{n-i} F_k^i \beta^{k(n-i)-ti+j} \; .$$

Substituting (4) for (3), and dividing by $\sqrt{5}$ gives:

(A) $$F_j F_{k+t}^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{ti} F_t^{n-i} F_k^i F_{k(n-i)-ti+j} \; ,$$

while adding (3) and (4) results in

(B) $$L_j F_{k+t}^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{ti} F_t^{n-i} F_k^i L_{k(n-i)-ti+j} \; .$$

We note that

$$F_{k(n-i)-ti+j}^2 = \frac{1}{5} \left(L_{2k(n-i)-2ti+2j} - 2(-1)^{k(n-i)-ti+j}\right) \; ,$$

$$L_{k(n-i)-ti+j}^2 = L_{2k(n-i)-2ti+2j} + 2(-1)^{k(n-i)-ti+j} \; ,$$

and that

(5) $$2(-1)^j\left[F_{2k}(-1)^t + F_{2t}(-1)^k\right]^n = \sum_{i=0}^{n}\binom{n}{i}F_{2t}^{n-i}F_{2k}^{i}\left[2(-1)^{k(n-i)-ti+j}\right] .$$

Substitute 2j, 2k, and 2t for j, k, and t in (B), and subtract (5) to get:

$$L_{2j}F_{2(k+t)}^n - 2(-1)^j\left[F_{2k}(-1)^t + F_{2t}(-1)^k\right]^n = \sum_{i=0}^{n}\binom{n}{i}F_{2t}^{n-i}F_{2k}^{i}F_{k(n-i)-ti+j}^2 .$$

We add the same equations to conclude that:

$$L_{2j}F_{2(k+t)}^n + 2(-1)^j\left[F_{2k}(-1)^t + F_{2t}(-1)^k\right]^n = \sum_{i=0}^{n}\binom{n}{i}F_{2t}^{n-i}F_{2k}^{i}L_{k(n-i)-ti+j}^2 .$$

These expressions may be simplified by observing that

$$\left[F_{2t}(-1)^t + F_{2t}(-1)^k\right]^n = (-1)^{tn}\left[F_{2k} + (-1)^{k-t}F_{2t}\right]^n ,$$

and that from the well-known identity

$$L_hF_g = F_{g+h} + (-1)^hF_{g-h} ,$$

it follows (by letting $g = k + t$ and $h = k - t$) that

$$F_{2k} + (-1)^{k-t}F_{2t} = L_{k-t}F_{k+t} .$$

Thus

(C) $$L_{2j}F_{2(k+t)}^n - 2(-1)^{j+tn}F_{k+t}^nL_{k-t}^n = \sum_{i=0}^{n}\binom{n}{i}F_{2t}^{n-i}F_{2k}^{i}F_{k(n-i)-ti+j}^2 ,$$

and

$$\text{(D)}\qquad L_{2j}F^n_{2(k+t)} + 2(-1)^{j+tn}F^n_{k+t}L^n_{k-t} = \sum_{i=0}^{n}\binom{n}{i}F^{n-i}_{2t}F^i_{2k}L^2_{k(n-i)-ti+j} .$$

We rewrite (1) and (2), using m in place of k:

$$\alpha^t F_m = F_{m+t} - \beta^m F_t \qquad \text{and} \qquad \beta^t F_m = F_{m+t} - \alpha^m F_t .$$

Therefore,

$$\alpha^{kt}F^k_m = \sum_{h=0}^{k}\binom{k}{h}(-1)^h F^{k-h}_{m+t}F^h_t \beta^{mh}$$

and

$$\beta^{kt}F^k_m = \sum_{h=0}^{k}\binom{k}{h}(-1)^h F^{k-h}_{m+t}F^h_t \alpha^{mh} .$$

Multiplying the first equation by α^{n-kt} and the second by β^{n-kt}, we get:

$$\text{(6)}\qquad \alpha^n F^k_m = \sum_{h=0}^{k}\binom{k}{h}(-1)^{h+n-kt}F^{k-h}_{m+t}F^h_t \beta^{mh-n+kt} ,$$

and

$$\text{(7)}\qquad \beta^n F^k_m = \sum_{h=0}^{k}\binom{k}{h}(-1)^{h+n-kt}F^{k-h}_{m+t}F^h_t \alpha^{mh-n+kt} .$$

We subtract (7) from (6) to get

$$F_nF_m^k = \sum_{h=0}^{k} \binom{k}{h} (-1)^{h+n-kt+1} F_{m+t}^{k-h} F_t^h F_{mn-n+kt}$$

or equivalently,

$$(-1)^{n+1}F_nF_m^k = (-1)^{kt} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_{m+t}^{k-h} F_t^h F_{mh-n+kt}$$

Adding (6) and (7), we get:

$$L_nF_m^k = \sum_{h=0}^{k} \binom{k}{h} (-1)^{h+n-kt} F_{m+t}^{k-h} F_t^h L_{mh-n+kt} ,$$

or

$$(-1)^n L_nF_m^k = (-1)^{kt} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_{m+t}^{k-h} F_t^h L_{mh-n+kt} .$$

Finally, we replace $(-1)^{n+1}F_n$ with F_{-n}; $(-1)^n L_n$ with L_{-n}; and $-n$ with n to obtain:

(E) $$F_nF_m^k = (-1)^{kt} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_{m+t}^{k-h} F_t^h F_{mh+n+kt}$$

(see [3]), and

(F) $$L_nF_m^k = (-1)^{kt} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_{m+t}^{k-h} F_t^h L_{mh+n+kt} .$$

As before, we observe that

$$F^2_{mh+n+kt} = \frac{1}{5}(L_{2mn+2n+2kt} - 2(-1)^{mh+n+kt}) \ ,$$

that

$$L^2_{mh+n+kt} = L_{2mh+2n+2kt} + 2(-1)^{mh+n+kt} \ ,$$

that

$$2(-1)^{n+kt}\left[F_{2(m+t)} + (-1)^{m+1}F_{2t}\right]^k = \sum_{h=0}^{k}\binom{k}{h}(-1)^h F^{k-h}_{2(m+t)}F^h_{2t}\left[2(-1)^{mh+n+kt}\right]$$

and that

$L_hF_g = F_{g+h} + (-1)^hF_{g-h} \Rightarrow$ (with $g = m + 2t;\ h = m$)$:F_{2(m+t)} + (-1)^{m+1}F_{2t} = L_{m+2t}F_m$.

We replace m, n and t in (F) with $2m$, $2n$ and $2t$ and perform the obvious subtraction and addition to obtain:

$$\text{(G)}\quad L_{2n}F^k_{2m} - 2(-1)^{n+kt}L^k_{m+2t}F^k_m = 5\sum_{h=0}^{k}\binom{k}{h}(-1)^hF^{k-h}_{2(m+t)}F^h_{2t}F^2_{mh+n+kt} \ ,$$

and

$$\text{(H)}\quad L_{2n}F^k_{2m} + 2(-1)^{n+kt}L^k_{m+2t}F^k_m = \sum_{h=0}^{k}\binom{k}{h}(-1)^hF^{k-h}_{2(m+t)}F^h_{2t}L^2_{mh+n+kt} \ .$$

Starting with $\alpha^m = AF_{m+k} + BL_m$.

By a procedure identical with that used to obtain (1) and (2), we get:

$$\text{(8)}\qquad \alpha^m L_k = \sqrt{5}F_{m+k} + \beta^k L_m \ ,$$

and

$$\beta^m L_k = -\sqrt{5}\,F_{m+k} + \alpha^k L_m \quad , \tag{9}$$

which lead to

$$\alpha^{mn+j} L_k^n = \sum_{i=0}^{n} \binom{n}{i} \sqrt{5}^{\,i} F_{m+k}^i L_m^{n-i} \beta^{k(n-i)} \alpha^j \tag{10}$$

and

$$\beta^{mn+j} L_k^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \sqrt{5}^{\,i} F_{m+k}^i L_m^{n-i} \alpha^{k(n-i)} \beta^j \quad . \tag{11}$$

Subtracting (11) from (10) and dividing by $\sqrt{5}$ gives

$$F_{mn+j} L_k^n = (-1)^j \sum_{i=0}^{n} \binom{n}{i} \sqrt{5}^{\,i-1} F_{m+k}^i L_m^{n-i} \left[\beta^{k(n-i)-j} - (-1)^i \alpha^{k(n-i)-j}\right]$$

or

$$F_{mn+j} L_k^n = (-1)^{j+1} \sum_{i=0}^{[n/2]} \binom{n}{2i} 5^i F_{m+k}^{2i} L_m^{n-2i} F_{k(n-2i)-j}$$

(I)

$$+ (-1)^j \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2i+1} 5^i F_{m+k}^{2i+1} L_m^{n-2i-1} L_{k(n-2i-1)-j} \quad ,$$

and adding (10) and (11) yields:

$$L_{mn+j}L_k^n = (-1)^j \sum_{i=0}^{n} \binom{n}{i} \sqrt{5}^i F_{m+k}^i L_m^{n-i}\left[\beta^{k(n-i)-j} + (-1)^i \alpha^{k(n-i)-j}\right],$$

or

$$L_{mn+j}L_k^n = (-1)^j \sum_{i=0}^{[n/2]} \binom{n}{2i} 5^i F_{m+k}^{2i} L_m^{n-2i} L_{k(n-2i)-j}$$

(J)

$$+ (-1)^{j+1} \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2i+1} 5^i F_{m+k}^{2i+1} L_m^{n-2i-1} F_{k(n-2i-1)-j} .$$

Equations (8) and (9) may be rewritten:

$$\sqrt{5}F_{m+k} = \alpha^m L_k - \beta^k L_m ,$$

and

$$\sqrt{5}F_{m+k} = -\beta^m L_k + \alpha^k L_m ,$$

which give

(12) $$\alpha^j \sqrt{5}^n F_{m+k}^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i L_k^{n-i} L_m^i \alpha^{m(n-i)+j} \beta^{ki} ,$$

and

(13) $$\beta^j \sqrt{5}^n F_{m+k}^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} L_k^{n-i} L_m^i \alpha^{ki} \beta^{m(n-i)+j} .$$

Adding (12) and (13), we get:

$$\sqrt{5}^n L_j F^n_{m+k} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{(k+1)i} L_k^{n-i} L_m^i \left[\alpha^{m(n-i)-ki+j} + (-1)^n \beta^{m(n-i)-ki+j} \right],$$

which, in turn, provides

(K) $$5^n L_j F^{2n}_{m+k} = \sum_{i=0}^{2n} \binom{2n}{i} (-1)^{(k+1)i} L_k^{2n-i} L_m^i L_{m(2n-i)-ki+j},$$

and

(L) $$5^n L_j F^{2n+1}_{m+k} = \sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{(k+1)i} L_k^{2n+1-i} L_m^i F_{m(2n+1-i)-ki+j}$$

We subtract (13) from (12) to get:

$$\sqrt{5}^{n+1} F_j F^n_{m+k} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{(k+1)i} L_k^{n-i} L_m^i \left[\alpha^{m(n-i)-ki+j} - (-1)^n \beta^{m(n-i)-ki+j} \right]$$

from which we get

(M) $$5^n F_j F^{2n}_{m+k} = \sum_{i=0}^{2n} \binom{2n}{i} (-1)^{(k+1)i} L_k^{2n-i} L_m^i F_{m(2n-i)-ki+j},$$

and

(N) $$5^{n+1} F_j F^{2n+1}_{m+k} = \sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{(k+1)i} L_k^{2n+1-i} L_m^i L_{m(2n+1-i)-ki+j}.$$

Once again, we note that

(14)
$$2(-1)^j\left[L_{2k} - (-1)^{k-m}L_{2m}\right]^{2n} = \sum_{i=0}^{2n}\binom{2n}{i}(-1)^i L_{2k}^{2n-i}L_{2m}^i\left[2(-1)^{m(2n-i)-ki+j}\right]$$

In (K), we let j, k, and m be replaced by 2j, 2k, and 2m and subtract (M):

(15)
$$5^n L_{2j}F_{2(m+k)}^{2n} - 2(-1)^j\left[L_{2k} - (-1)^{k-m}L_{2m}\right]^{2n} = 5\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i L_{2k}^{2n-i}L_{2m}^i F_{m(2n-i)-ki+j}^2 .$$

The corresponding addition provides

(16)
$$5^n L_{2j}F_{2(m+k)}^{2n} + 2(-1)^j\left[L_{2k} - (-1)^{k-m}L_{2m}\right]^{2n} = \sum_{i=0}^{2n}\binom{2n}{i}(-1)^i L_{2k}^{2n-i}L_{2m}^i L_{m(m-i)-ki+j}^2 .$$

Since

$$5F_{k+m}F_{k-m} = (\alpha^{k+m} - \beta^{k+m})(\alpha^{k-m} - \beta^{k-m}) = \alpha^{2k} - (\alpha\beta)^{k-m}(\alpha^{2n} + \beta^{2m}) + \beta^{2k} ,$$

or

(17)
$$5F_{k+m}F_{k-m} = L_{2k} - (-1)^{k-m}L_{2m} ,$$

we can rewrite (15) and (16):

(P)
$$5^{n-1}L_{2j}F_{2(m+k)}^{2n} - 2\cdot 5^{2n-1}(-1)^j F_{k+m}^{2n}F_{k-m}^{2n} = \sum_{i=0}^{2n}\binom{2n}{i}(-1)^i L_{2k}^{2n-i}L_{2m}^i F_{m(2n-i)-ki+j}^2 ,$$

and

(Q)
$$5^n L_{2j}F_{2(m+k)}^{2n} + 2\cdot 5^{2n}(-1)^j F_{k+m}^{2n}F_{k-m}^{2n} = \sum_{i=0}^{2n}\binom{2n}{i}(-1)^i L_{2k}^{2n-i}L_{2m}^i L_{m(2k-i)-ki+j}^2 .$$

We next observe that

$$2(-1)^{m+j}\left[L_{2k} - (-1)^{k-m}L_{2m}\right]^{2n+1}$$

$$= \sum_{i=0}^{2n+1} \binom{2n+1}{i}(-1)^i L_{2k}^{2n+i-j} L_{2m}^i \left[2(-1)^{m(2n+1-i)-ki+j}\right]$$

and again we employ (17) and treat (N) as we did (K) to conclude

$$\text{(R)} \quad 5^{n+1}F_{2j}F_{2(m+k)}^{2n+1} + 2\cdot 5^{2n+1}(-1)^{m+j}F_{k+m}^{2n+1}\,F_{k-m}^{2n+1}$$

$$= \sum_{i=0}^{2n+1} \binom{2n+1}{i}(-1)^i L_{2k}^{2n+1-i} L_{2m}^i L^2_{m(2n+1-i)-ki+j}$$

and

$$\text{(S)} \quad 5^{n}F_{2j}F_{2(m+k)}^{2n+1} - 2\cdot 5^{2n}(-1)^{m+j}F_{k+m}^{2n+1}\,F_{k=m}^{2n+1}$$

$$= \sum_{i=0}^{2n+1} \binom{2n+1}{i}(-1)^i L_{2k}^{2n+1-i} L_{2m}^i F^2_{m(2n+1-i)-ki+j} \quad .$$

Starting with

$$\text{(18)} \qquad \alpha^m = AL_{m+k} + BL_m \ ,$$

we get

$$L_{m+k} = \sqrt{5}\,\alpha^m F_k + \beta^k L_m \ .$$

Interchanging variables does not produce a second useful equation. However,

$$\text{(19)} \qquad \beta^m = A'L_{m+k} + B'L_m$$

yields

$$L_{m+k} = -\sqrt{5}\,\beta^m F_k + \alpha^k L_m .$$

Proceeding as usual, we get

$$\alpha^j L_{m+k}^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{(m+1)i} \sqrt{5}^{\,i} L_m^{n-i} F_k^i \alpha^{k(n-i)-mi+j} ,$$

and

$$\beta^j L_{m+k}^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{mi} \sqrt{5}^{\,i} L_m^{n-i} F_k^i \beta^{k(n-i)-mi+j} .$$

Adding,

$$L_j L_{m+k}^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{mi} \sqrt{5}^{\,i} L_m^{n-i} F_k^i \left[(-1)^i \alpha^{k(n-i)-mi+j} + \beta^{k(n-i)-mi+j}\right]$$

or equivalently,

$$L_j L_{m+k}^n = \sum_{i=0}^{[n/2]} \binom{n}{2i} 5^i L_m^{n-2i} F_k^{2i} L_{k(n-2i)-2mi+j}$$

(T)

$$+ (-1)^m \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2i+1} 5^{i+1} L_m^{n-2i-1} F_k^{2i+1} F_{k(n-2i-1)-m(2i+1)+j}$$

and subtracting,

$$\sqrt{5}\, F_j L_{m+k}^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{mi} \sqrt{5}^{\,i} L_m^{n-i} F_k^i \left[(-1)^i \alpha^{k(n-i)-mi+j} - \beta^{k(n-i)-mi+j}\right]$$

or

$$F_jL_{m+k}^n = \sum_{i=0}^{[n/2]} \binom{n}{i} 5^i L_m^{n-2i} F_k^{2i} F_{k(n-2i)-m(2i)+j}$$

(U)

$$+ (-1)^m \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2i+1} 5^i L_m^{n-2i-1} F_k^{2i+1} L_{k(n-2i-1)-m(2i+1)+j} .$$

We rewrite (18) and (19) and proceed as before:

$$L_m\alpha^k = L_{m+k} + \sqrt{5}B^m F_k$$

and

$$L_m\beta^k = L_{m+k} - \sqrt{5}\alpha^m F_k$$

yield

$$\alpha^{kn+j}L_m^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^j \sqrt{5}^i L_{m+k}^{n-i} F_k^i \beta^{mi-j}$$

and

$$\beta^{kn+j}L_m^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i+j} \sqrt{5}^i L_{m+k}^{n-i} F_k^i \alpha^{mi-j} .$$

We add, to give

$$L_{kn+j}L_m^n = (-1)^j \sum_{i=0}^{n} \binom{n}{i} \sqrt{5}^i L_{m+k}^{n-i} F_k^i \left[\beta^{mi-j} + (-1)^i \alpha^{mi-j}\right] ,$$

or

(V)
$$L_{kn+j}L_m^n = (-1)^j \sum_{i=0}^{[n/2]} \binom{n}{2i} 5^i L_{m+k}^{n-2i} F_k^{2i} L_{2mi-j}$$
$$+ (-1)^{j+i} \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2i+1} 5^{i+1} L_{m+k}^{n-2i-1} F_k^{2i+1} F_{m(2i+1)-j}$$

and subtract, for

$$\sqrt{5}\, F_{kn+j}L_m^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^j \sqrt{5}^{\,i} L_{m+k}^{n-i} F_k^i \left[\beta^{mi-j} - (-1)^i \alpha^{mi-j}\right]$$

or

(W)
$$F_{kn+j}L_m^n = (-1)^{j+1} \sum_{i=0}^{[n/2]} \binom{n}{2i} 5^i L_{m+k}^{n-2i} F_k^{2i} F_{2mi-j}$$
$$+ (-1)^j \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2i+1} 5^i L_{m+k}^{n-2i-1} F_k^{2i+1} L_{m(2i+1)-j} \quad .$$

3. EXTENSION TO FIBONACCI AND LUCAS POLYNOMIALS

The Fibonacci polynomials $\{f_n(x)\}$ are defined by:

$$f_1(x) = 1; \quad f_2(x) = x; \quad f_{n+2}(x) = xf_{n+1}(x) + f_n(x) \ .$$

The Lucas polynomials are similarly defined:

$$\ell_1(x) = x; \quad \ell_2(x) = x^2 + 2; \quad \ell_{n+2}(x) = x\ell_{n+1}(x) + \ell_n(x) \ .$$

Let λ_1 and λ_2 be the roots of $\lambda^2 = x\lambda + 1$;

$$\lambda_1(x) = \frac{1}{2}(x + \sqrt{x^2 + 4}); \qquad \lambda_2(x) = \frac{1}{2}(x - \sqrt{x^2 + 4}) .$$

It is easily verified that:

$$f_n(x) = (\lambda_1^n(x) - \lambda_2^n(x))/(\lambda_1(x) - \lambda_2(x))$$

and

$$\ell_n(x) = \lambda_1^n(x) + \lambda_2^n(x) .$$

In view of the striking similarities between the Binet forms of the Fibonacci and Lucas polynomials, and the corresponding forms for the Fibonacci and Lucas sequences, it is hardly surprising that there exists an identity involving $\lambda_1(x)$, $\lambda_2(x)$, $f_n(x)$ and $\ell_n(x)$ paralleling each identity involving α, β, F_n, and L_n. For example, corresponding to (A), we get:

$$\text{(19')} \qquad f_i(x)f^n_{k+t}(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{ti} f_t^{n-i}(x) f_k^i(x) f_{kn+j-(k+t)i}(x) ,$$

and, corresponding to (E), we have:

$$\text{(E')} \qquad f_n(x)f_m^k(x) = (-1)^{kt} \sum_{h=0}^{k} \binom{k}{h} (-1)^h f^{k-h}_{m+t} f_t^h(x) f_{mn+n+kt}(x)$$

In fact, the identities (A) through (W) are special cases of the Fibonacci-Lucas polynomial identities, obtained by setting $x = 1$.

One observes that $f_n(2)$ obeys: $C_{n+2} = 2C_{n+1} + C_n$; $C_0 = 0$, $C_1 = 1$. This sequence is the Pell sequence. Since

$$\ell_n(x) = f_{n+1}(x) + f_{n-1}(x) ,$$

one can define

$$\ell_n(2) = C_n^* = C_{n+1} + C_{n-1}$$

to make complete substitutions in identities (A)-(W).

4. A FURTHER EXTENSION

Let $g_n(x)$ obey $g_{n+2}(x) = xg_{n+1}(x) - g_n(x)$; $g_0(x) = 0$; $g_1(x) = 1$. Then

$$\begin{aligned} g_n(x) &= 1/(\sqrt{x^2+4})\{[(x+\sqrt{x^2+4})/2]^n - [(x-\sqrt{x^2+4})/2]^n\} \\ &= (\lambda_1^n - \lambda_2^n)/(\lambda_1 - \lambda_2)\ , \end{aligned}$$

where λ_1 and λ_2 are roots of $\lambda_2 - x\lambda + 1 = 0$. Also, let

$$h_n(x) = \lambda_1^n + \lambda_2^n = g_{n+1}(x) - g_{n-1}(x)\ .$$

These sequences of polynomials are simply related to the Chebychev polynomials of the first and second kind.

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