ELEMENTARY PROBLEMS AND SOLUTIONS

Edited By
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-208 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n.$$

Prove both of the following and generalize:

(a) $$F^2_{n+2} = 3F^2_{n+1} - F^2_n - 2(-1)^n$$

(b) $$L^2_{n+2} = 3L^2_{n+1} - L^2_n + 10(-1)^n \; .$$

B-209 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Do the analogue of B-208 for the Pell sequence defined by

$$P_0 = 0, \quad P_1 = 1, \quad P_{n+2} = 2P_{n+1} + P_n, \quad \text{and} \quad Q_n = P_n + P_{n-1} \, .$$

B-210 Proposed by Guy A. R. Guillotte, Montreal, Quebec, Canada.

Let $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Prove that $S > 803/240$, where

$$S = \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \cdots \quad .$$

B-211 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let F_n be the n^{th} term in the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$. Solve the recurrence

$$D_{n+1} = 2D_n + F_{2n+1}$$

subject to the initial conditions $D_1 = 1$ and $D_2 = 3$.

B-212 Proposed by Tomas Djerverson, Albrook College, Tigertown on the Rio.

Give examples of interesting functions f and g such that

$$f(m,n) = g(m + n) - g(m) - g(n) \, .$$

(One example is $f(m,n) = mn$ and $g(n) = \binom{n}{2} = n(n - 1)/2$.)

B-213 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Given n points on a straight line, find the number of subsets (including the empty set) of the n points in which consecutive points are not allowed. Also find the corresponding number when the points are on a circle.

SOLUTIONS

A SIXTY-ORDER FIBONACCI-LUCAS IDENTITY

B-190 A repeat of B-186 with a typographical error corrected.

Let L_n be the n^{th} Lucas number and show that

$$L_{5n}/L_n = [L_{2n} - 3(-1)^n]^2 + (-1)^n 25F_n^2 \, .$$

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Since L_{5n} and

$$\{[L_{2n} - 3(-1)^n]^2 + (-1)^n 25 F_n^2\} L_n$$

satisfy the same sixth-order linear homogeneous recurrence, the result is proved by verifying it for $n = -2, -1, 0, 1, 2,$ and 3 (and then relying on mathematical induction).

Also solved by W. C. Barley, Wray G. Brady, Warren Cheves, Herta T. Freitag, Jo Carol Gordon, John A. Hitchcock, Edgar Karst, Bob Topley, Andrew Wyatt, Rev. Robert Zuparko, and the Proposer.

THE HUNTER UNVEILED

B-191 Proposed by Guy A. R. Guillotte, Montreal, Quebec, Canada.

In this alphametic, each letter represents a particular but different digit, all ten digits being represented here. It must only be that well-known mathematical teaser from Toronto, J. A. H. Hunter, but what is the value of HUNTER?

$$\begin{array}{r} \text{MR} \\ \text{HUNTER} \\ \text{MADE} \\ \underline{\text{A}} \\ \text{TEASER} \end{array}$$

Solution by David Zeitlin, Minneapolis, Minnesota.

The value of HUNTER is 198207, where the unique solution is given by

$$\begin{array}{r} 57 \\ 198207 \\ 5340 \\ \underline{3} \\ 203607 \end{array}$$

We note that

(1) $$2R + E + A = 10C_1 + R ,$$

(2) $$C_1 + M + E + D = 10C_2 + E ,$$

(3) $$C_2 + T + A = 10C_3 + S ,$$

(4) $$C_3 + N + M = 10C_4 + A ,$$

(5) $$C_4 + U = 10C_5 + E ,$$

and

(6) $$C_5 + H = T,$$

where C_i, $i = 1, 2, \cdots, 5$, are carry-overs from the i^{th} column. Since $C_5 = 1$, we find from (5) that $C_4 = 1$, with $U = 9$ and $E = 0$; and thus, from (1), that $C_1 = 1$. From (2), $C_2 = 1$, and from (3), $C_3 = 0$ or 1. All cases for $C_3 = 1$ are non-solutions. For $C_3 = 0$, the single solution is obtained when

$$(A,D,E,H,M,N,R,S,T,U) = (3,4,0,1,5,8,7,6,2,9) .$$

Also solved by W. C. Barley, Wray G. Brady, Albert Gommel, Jo Carol Gordon, J. A. H. Hunter, Edgar Karst, John W. Milsom, C. B. A. Peck, Darla Perry, Azriel Rosenfeld, and the Proposer.

A FOURTH-ORDER F-L IDENTITY

B-192 Proposed by Warren Cheves, Littleton, North Carolina.

Prove that $F_{3n} = L_n F_{2n} - (-1)^n F_n$.

Solution by Herta T. Freitag, Hollins, Virginia.

One needs to show that

$$\alpha^{3n} - \beta^{3n} = (\alpha^n + \beta^n)(\alpha^{2n} - \beta^{2n}) - (-1)^n(\alpha^n - \beta^n) ,$$

where α and β are $(1 \pm \sqrt{5})/2$.

This, however, is immediately seen by using the relationship $\alpha\beta = -1$, and simplifying.

Also solved by W. C. Barley, Wray G. Brady, Mike Franusich, Jo Carol Gordon, John A. Hitchcock, Stu Hobbs, Edgar Karst, John Kegel, Scott King, John W. Milsom, C. B. A. Peck, Darla Perry, Patricia Shay, Don C. Stevens, Bob Tepley, Andrew Wyatt, David Zeitlin, Rev. Robert Zuparko, and the Proposer.

ANOTHER F-L IDENTITY

B-193 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that $L_{n+p} \pm L_{n-p}$ is $5F_pF_n$ or L_pL_n, depending on the choice of sign on whether p is even or odd.

Solution by John Kegel, Fort Lauderdale, Florida.

It is a well-known fact that

$$F_n = \frac{a^n - b^n}{a - b} \tag{1}$$

and

$$L_n = a^n + b^n \,, \tag{2}$$

where

$$a = \frac{1}{2}(1 + \sqrt{5})$$

and

$$b = \frac{1}{2}(1 - \sqrt{5}) \,,$$

which also gives

(3) $$ab = -1$$

and

(3') $$(a - b)^2 = 5 .$$

Now

$$\begin{aligned} L_p L_n &= (a^p + b^p)(a^n + b^n) \\ &= a^{n+p} + b^{n+p} + a^p b^n + b^p a^n \\ &= L_{n+p} + (ab)^p (a^{n-p} + b^{n-p}) ; \end{aligned}$$

Thus

(4) $$L_p L_n = L_{n+p} + (-1)^p L_{n-p} .$$

Likewise,

$$\begin{aligned} 5 F_p F_n &= 5\left(\frac{a^p - b^p}{a - b}\right)\left(\frac{a^n - b^n}{a - b}\right) \\ &= \frac{5}{(a - b)^2} (a^{n+p} + b^{n+p} - a^p b^n - a^n b^p) \\ &= \frac{5}{5} [L_{n+p} - (a^p b^p)(a^{n-p} + b^{n-p})] ; \end{aligned}$$

Thus

(5) $$5 F_p F_n = L_{n+p} - (-1)^p L_{n-p} ,$$

Hence (4) and (5) give

(6) $$L_{n+p} \pm L_{n-p} = 5 F_p F_n \quad \text{or} \quad L_p L_n \quad (p \text{ odd or even})$$

and the proof is complete.

Also solved by W. C. Barley, Wray G. Brady, Herta T. Freitag, Jo Carol Gordon, David Zeitlin, and the Proposer.

SECOND ORDER IN n, FIFTH IN k

B-194 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico. (Corrected Statement).

Show that

$$L_{n+4k} - L_n = 5F_k\,[F_{n+3k} + (-1)^k F_{n+k}]\ .$$

Solution by C. B. A. Peck, State College, Pennsylvania.

From results of Brother Alfred (Fibonacci Quarterly, Vol. 1, No. 4, p. 55), or H. H. Ferns (Fibonacci Quarterly, Vol. VII, No. 1, p. 1), we have

$$5F_uF_v = [L_{u+v} - (-1)^v L_{u-v}]\ .$$

Replace v by k and u successively by n + 3k and n + k. Multiply the second of these identities by $(-1)^k$ and add to the first; this gives the (corrected) desired result.

Also solved and corrected by Wray G. Brady, Herta T. Freitag, John Kegel, David Zeitlin, and the Proposer. The error in the statement was also noted by W. C. Barley.

GENERALIZED FIBONOMIALS

B-195 Proposed by David Zeitlin, Minneapolis, Minnesota.

Let $\left\{{n \atop r}\right\}$ denote $L_nL_{n-1}\cdots L_{n-r+1}/L_1L_2\cdots L_r$. Show that

$$L_n^3/6 = \left\{{n+2 \atop 3}\right\} - 2\left\{{n+1 \atop 3}\right\} - \left\{{n \atop 3}\right\}\ .$$

Solution by A. K. Gupta, University of Arizona, Tuscon, Arizona.

From formula (2) (on p. 447 of Fibonacci Quarterly, Vol. 8, No. 4) of the Proposer's solution to B-176, we have

$$\text{(A)} \qquad 2H^3 = H_1H_2H_3\left\{\begin{bmatrix} n+2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} n+1 \\ 3 \end{bmatrix} + \begin{bmatrix} n \\ 3 \end{bmatrix}\right\},$$

where H_n satisfies

$$H_{n+2} = H_{n+1} + H_n$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix} = H_nH_{n-1} \cdots H_{n-r+1}/H_1H_2 \cdots H_r .$$

The desired result is obtained from (A) for $H_n + L$

$$H_n + L_n, \quad \begin{Bmatrix} n \\ r \end{Bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix},$$

and

$$H_1H_2H_3 = L_1L_2L_3 = 2\cdot 1\cdot 3 = 6 \quad .$$

Also solved by W. C. Barley, Wray G. Brady, Warren Cheves, Herta T. Freitag, John Kegel, John W. Milsom, and the Proposer.