SOME SPECIAL FIBONACCI AND LUCAS GENERATING FUNCTIONS

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In [1], Hoggatt and Bicknell derived by matrix methods that

$$\sum_{i=0}^{2n+2} \binom{2n+2}{i} F_i^2 = 5^n L_{2n+2}$$

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_i^2 = 5^n F_{2n+1}$$

We next list three more similar sums.

(a) $$\sum_{k=0}^{n} \binom{n}{k} F_k = 1^n F_{2n}$$

(b) $$\sum_{k=0}^{n} \binom{n}{k} F_{3k} = 2^n F_{2n}$$

(c) $$\sum_{k=0}^{n} \binom{n}{k} F_{4k} = 3^n F_{2n} \quad .$$

Identity (a) is well known, while (b) was in a private communication from D. Lind, and (c) is a special case of Problem B-88 in the Fibonacci Quarterly, April, 1966, p. 149.

In [2], various special related results are also derived by matrix methods. Here, we derive a new class of generating functions by following the suggestion given in [3]. The column generators for Pascal's left-adjusted triangle are

$$g_n(x) = \frac{x^n}{(1 - x)^{n+1}}, \qquad n = 0, 1, 2, \cdots ,$$

while the generating function for the Fibonacci numbers is

$$G(x) = \frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n .$$

If we now sum

$$\sum_{n=0}^{\infty} F_n g_n(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} F_k x^n$$

$$= \frac{1}{1 - x} \sum_{n=0}^{\infty} F_n \left(\frac{x}{1 - x}\right)^n$$

$$= \frac{1}{1 - x} \frac{\frac{x}{1 - x}}{1 - \left(\frac{x}{1 - x}\right) - \left(\frac{x}{1 - x}\right)^2}$$

$$= \frac{x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n} x^n .$$

Thus

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n} .$$

Now, if we sum

$$\sum_{n=0}^{\infty} F_{2n} g_n(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} F_{2k} \right) x^n = \frac{x}{1 - 5x + 5x^2} .$$

This is a special case of the general class of identities

$$\frac{L_m x}{1 - 5F_m x + (-1)^{m+1} 5x^2} = L_m x + 5F_{2m} x^2 + 5L_{3m} x^3 + 5^2 F_{4m} x^4$$
$$+ 5^2 L_{5m} x^5 + \cdots + 5^k F_{2km} x^{2k}$$
$$+ 5^k L_{(2k+1)m} x^{2k+1} + \cdots .$$

We discuss first a related special case. To see this requires a few identities and a neat trick in algebra. It is easy to establish that

$$\frac{3 - 2x}{1 - 3x + x^2} = \sum_{k=0}^{\infty} L_{2k+2} x^k$$

$$\frac{x - x^2}{1 - 3x + x^2} = \sum_{k=0}^{\infty} F_{2k+1} x^k .$$

Now,

$$\frac{3x^2 - 10x^4}{1 - 15x^2 + 25x^4} = \sum_{k=0}^{\infty} L_{2k+2} x^{2k+2} 5^k$$

$$\frac{x(1 - 5x^2)}{1 - 15x^2 + 25x^4} = \sum_{k=0}^{\infty} F_{2k+1} x^{2k+1} 5^k .$$

Notice that

$$\frac{x + 3x^2 - 5x^3 - 10x^4}{1 - 15x^2 + 25x^4} = \frac{(x - 2x^2)(1 + 5x + 5x^2)}{(1 - 5x + 5x^2)(1 + 5x + 5x^2)} = \frac{x - 2x^2}{1 - 5x + 5x^2} .$$

Next, we need

$$\frac{x(1 - x)}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_n^2 x^n .$$

Summing as before,

$$\sum_{n=0}^{\infty} F_n^2 g_n(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_k^2\right)x^n$$

$$= \sum_{n=0}^{\infty} F_n^2 \frac{x^n}{(1 - x)^{n+1}} = \frac{1}{1 - x}\sum_{n=0}^{\infty} F_n^2\left(\frac{x}{1 - x}\right)^n$$

$$= \frac{1}{1 - x}\,\frac{\frac{x}{1 - x}\left(1 - \frac{x}{1 - x}\right)}{1 - 2\left(\frac{x}{1 - x}\right) - 2\left(\frac{x}{1 - x}\right)^2 + \left(\frac{x}{1 - x}\right)^3}$$

$$= \frac{x - 2x^2}{1 - 5x + 5x^2} .$$

Thus

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}F_k^2\right]x^n = \frac{x - 2x^2}{1 - 5x + 5x^2} = \sum_{k=0}^{\infty} 5^k(F_{2k+1} + xL_{2k+2})x^{2k+1}$$

and

$$\sum_{k=0}^{2n+2}\binom{2n + 2}{k}F_k^2 = 5^n L_{2n+2}$$

$$\sum_{k=0}^{2n+1}\binom{2n + 1}{k}F_k^2 = 5^n F_{2n+1} ,$$

which are given in the first paragraph of the paper. Clearly, then, the

$$\sum_{k=0}^{n} \binom{n}{k} F_k^2 \qquad \text{and} \qquad \sum_{k=0}^{n} \binom{n}{k} F_{2k}$$

are related. We return now to the special case

$$\frac{x}{1 - 5x + 5x^2} = L_1 x + F_2 5x^2 + L_3 5x^3 + \cdots .$$

To see this, we write

$$\frac{x + x^2}{1 - 3x + x^2} = \sum_{k=0}^{\infty} L_{2k+1} x^{k+1}$$

$$\frac{x}{1 - 3x + x^2} = \sum_{k=0}^{\infty} F_{2k} x^k .$$

Next,

$$\frac{x(1 + 5x^2)}{1 - 15x^2 + 25x^4} = \sum_{k=0}^{\infty} L_{2k+1} 5^k x^{2k+1}$$

and

$$\frac{5x^2}{1 - 15x^2 + 25x^4} = \sum_{k=0}^{\infty} F_{2k} 5^k x^{2k} .$$

Thus,

$$\frac{x(1 + 5x + 5x^2)}{1 - 15x^2 + 25x^4} = \sum_{k=0}^{\infty} 5^k(F_{2k} + xL_{2k+1})x^{2k} .$$

But,

$$\begin{aligned} 1 - 15x^2 + 25x^4 &= 1 + 10x^2 + 25x^4 - 25x^2 = (1 + 5x^2)^2 - (5x)^2 \\ &= (1 + 5x + 5x^2)(1 - 5x + 5x^2) . \end{aligned}$$

Thus,

$$\frac{x}{1 - 5x + 5x^2} = \sum_{k=0}^{\infty} 5^k(F_{2k} + xL_{2k+1})x^{2k}$$

and

$$\sum_{k=0}^{2n} \binom{2n}{k} F_{2k} = 5^n F_{2n}$$

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_{2k} = 5^n L_{2n+1} .$$

We now return to our general class of identities

$$\frac{L_m x}{1 - 5F_m x + (-1)^{m+1} 5x^2} = \sum_{k=0}^{\infty} 5^k(F_{2km} + xL_{(2k+1)m})x^{2k} .$$

We begin by writing

$$\frac{xL_m(1 + (-1)^{m+1}x^2)}{1 - L_{2m}x + x^2} = \sum_{k=0}^{\infty} L_{(2k+1)m}\, x^{k+1}$$

while

$$\frac{xL_m(1 + (-1)^{m+1}5x^2)}{1 - 5L_{2m}x^2 + 25x^4} = \sum_{k=0}^{\infty} L_{(2k+1)m}5^k x^{2k+1} .$$

Next,

$$\frac{F_{2m}5x^2}{1 - 5L_{2m}x + 25x^2} = \sum_{k=0}^{\infty} F_{2km}5^k x^{2k} ,$$

$$\frac{xL_m(1 + (-1)^{m+1}5x^2) + xL_m(F_m 5x)}{1 - 5L_{2m}x^2 + 25x^4} = \sum_{k=0}^{\infty} 5^k (F_{2km} + xL_{(2k+1)m})x^{2k} ;$$

$$\frac{xL_m(1 + 5xF_m + (-1)^{m+1}5x^2)}{1 - 5L_{2m}x^2 + 25x^4} = \frac{xL_m}{1 - 5F_m x + (-1)^{m+1}5x^2}$$

since

$$\begin{aligned} 5L_{2m} = 5L_m^2 + 10(-1)^{m+1} &= 5(5F_m^2 + 4(-1)^m + 2(-1)^{m+1}) \\ &= 25F_m^2 - 10(-1)^{m+1} . \end{aligned}$$

Thus

$$1 - 5L_{2m}x^2 + 25x^4 = 1 + 10(-1)^{m+1}x^2 + 25x^4 - 25F_m^2 x^2$$

or

$$\begin{aligned} (1 - 5L_{2m}x^2 + 25x^4) &= (1 + 5(-1)^{m+1}x^2)^2 - 25F_m^2 x^2 \\ &= (1 - 5F_m x + (-1)^{m+1}5x^2)(1 + 5F_m x + 5(-1)^{m+1}x^2). \end{aligned}$$

We now return to the general problem.

Remember,

$$\frac{L_m x}{1 - 5F_m x + (-1)^{m+1} 5x^2} = \sum_{k=0}^{\infty} 5^k (F_{2km} + xL_{(2k+1)m})x^{2k} .$$

We start with the general problem. The generating function for every m^{th} Fibonacci number is

$$\frac{F_m x}{1 - L_m x + (-1)^m x^2} = \sum_{k=0}^{\infty} F_{km} x^k .$$

Consider the sum

$$\sum_{n=0}^{\infty} F_{mn} g_n(x) = \frac{1}{1-x} \sum_{n=0}^{\infty} F_{mn} \left(\frac{x}{1-x}\right)^n$$

$$= \frac{1}{1-x} \frac{F_m \frac{x}{1-x}}{1 - L_m \left(\frac{x}{1-x}\right) + (-1)^m \left(\frac{x}{1-x}\right)^2}$$

$$= \frac{F_m x}{(1-x)^2 - L_m x(1-x) + (-1)^m x^2}$$

$$= \frac{F_m x}{1 - (L_m + 2)x + (L_m + 1 + (-1)^m)x^2}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} F_{km} \right] x^n .$$

Now, from

$$L_m^2 = L_{2m} + 2(-1)^m$$

and

$$L_n^2 - 5F_n^2 = 4(-1)^n$$

one can obtain four useful identities:

$$L_{4m} + 2 = L_{2m}^2$$

$$L_{4m} - 2 = L_{2m}^2 - 4 = 5F_{2m}^2$$

$$L_{4m+2} + 2 = L_{2m+1}^2 + 4 = 5F_{2m+1}^2$$

$$L_{4m+2} - 2 = L_{2m+1}^2 .$$

Thus, for $m = 2a$ (even)

$$\frac{F_{2s}x}{1 - (L_{2s} + 2)x + (L_{2s} + 2)x^2} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}F_{2sk}\right]x^n .$$

We now discuss two special cases.

(A)
$$\frac{F_{4m}x}{1 - L_{2m}^2 x + L_{2m}^2 x^2} = \frac{F_{2m}(L_{2m}x)}{1 - L_{2m}(L_{2m}x) + (L_{2m}x)^2}$$

$$= \sum_{n=0}^{\infty} F_{2mn}(L_{2m}x)^n$$

$$= \sum_{n=0}^{\infty} F_{2mn}L_{2m}^n x^n .$$

Thus,

$$\sum_{k=0}^{n} \binom{n}{k} F_{4mk} = L_{2m}^{n} F_{2mn} .$$

This is Problem H-88, Fibonacci Quarterly, April 1966, p. 149.

(B)
$$\frac{F_{4m+2}\,x}{1 - (L_{4m+2} + 2)x + (L_{4m+2} + 2)x^2}$$

$$= \frac{L_{2m+1}(F_{2m+1}\,x)}{1 - 5F_{2m+1}(F_{2m+1}\,x) + 5(F_{2m+1}\,x)^2}$$

$$= \frac{L_{2m+1}\,y}{1 - 5F_{2m+1}\,y + 5y^2}$$

$$= \sum_{k=0}^{\infty} 5^k (F_{(4m+2)k} + y L_{(2k+1)(2m+1)})y^{2k}$$

$$= \sum_{k=0}^{\infty} 5^k (F_{(4m+2)k} + F_{2m+1} L_{(2k+1)(2m+1)}) F_{2m+1}^{2k}\, x^{2k} .$$

Thus,

$$\sum_{k=0}^{2n} \binom{2n}{k} F_{(4m+2)k} = 5^n F_{(4m+2)n}\, F_{2m+1}^{2n}$$

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_{(4m+2)k} = 5^n L_{(2n+1)(2m+1)}\, F_{2m+1}^{2n+1} .$$

Suppose, on the other hand, that we wish to alternate the signs in the above sums. Consider the sums

$$\sum_{m=0}^{\infty} F_{km}(-1)^m g_m(x) = \frac{1}{1-x}\,\frac{F_m \frac{-x}{1-x}}{1 - L_m \frac{-x}{1-x} + (-1)^m\left(\frac{-x}{1-x}\right)^2}$$

$$= -\frac{F_m x}{(1-x)^2 + L_m x(1-x) + (-1)^m x^2}$$

$$= \frac{-F_m x}{1 + (L_m - 2)x - (L_m - 1 - (-1)^m)x^2}$$

again for even m. Thus,

$$\frac{-F_{4m} x}{1 + 5F_{2m}^2 x + 5F_{2m}^2 x^2} = \frac{L_{2m}(-F_{2m} x)}{1 - 5F_{2m}(-F_{2m} x) + 5(-F_{2m} x)^2}$$

$$= \frac{L_{2m} y}{1 - 5F_{2m} y + 5y^2}$$

$$= \sum_{k=0}^{\infty} 5^k (F_{2mk} + y L_{(2k+1)2m}) y^{2k}$$

$$= \sum_{k=0}^{\infty} 5^k (F_{2mk} - F_{2m} L_{(2k+1)2m}) F_{2m}^{2k} x^{2k} .$$

Thus,

$$\sum_{k=0}^{2n} (-1)^{2n+k} \binom{2n}{k} F_{4mk} = F_{2m}^{2n} F_{2mn}$$

$$\sum_{k=0}^{2n+1} (-1)^{2n+1+k} \binom{2n+1}{k} F_{4mk} = F_{2m}^{2n+1} L_{(2k+1)(2n+1)} .$$

Proceeding similarly with

$$\frac{F_{2m+1}(-L_{2m+1}x)}{1 - L_{2m+1}(-L_{2m+1}x) - (-L_{2m+1}x)^2} = \frac{F_{2m+1}y}{1 - L_{2m+1}y - y^2}$$

$$= \sum_{k=0}^{\infty} F_{(2m+1)k}\, y^k$$

$$= \sum_{k=0}^{\infty} (-1)^k L_{2m+1}^k F_{(2m+1)k}\, x^k .$$

Thus

$$\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} F_{(4m+2)k} = L_{2m+1}^n F_{(2m+1)n} .$$

There remains unsolved

$$\sum_{k=0}^{n} \binom{n}{k} F_k^m \qquad \text{and} \qquad \sum_{k=0}^{n} \binom{n}{k} F_{mk}$$

for m odd and greater than 3. Corresponding formulas are given also in [4] as follows:

$$\sum_{k=0}^{2n} \binom{2n}{k} L_{(4m+2)k} = 5^n L_{(2m+1)2n} F_{2m+1}^{2n}$$

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} L_{(4m+2)k} = 5^n F_{(2m+1)(2n+1)} F_{2m+1}^{2n+1}$$

$$\sum_{k=0}^{2n} \binom{n}{k} (-1)^{n+k} L_{(4m+2)k} = L_{(2m+1)n} L_{2m+1}^{n}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^{k} L_{4mk} = 5^{n} L_{4mn} F_{2m}^{2n}$$

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^{k+1} L_{4mk} = 5^{n} F_{2m(2n+1)} F_{2m}^{2n+1} .$$

REFERENCES

1. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Some New Fibonacci Identities," Fibonacci Quarterly, Vol. 2, No. 1, Feb. 1964, pp. 29-32.
2. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Matrix Generation of Fibonacci Identities for F_{2nk}," to appear, Fibonacci Quarterly.
3. V. E. Hoggatt, Jr., "A New Angle on Pascal's Triangle," Fibonacci Quarterly, Vol. 6, No. 4, Oct. 1968, pp. 221-234.
4. John Wessner, "Binomial Sums of Fibonacci Powers," Fibonacci Quarterly, Vol. 4, No. 4, Dec. 1966, pp. 355-358.
5. H. L. Leonard, Jr., Fibonacci and Lucas Identities and Generating Functions, San Jose State College Master's Thesis, 1969.

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SOME FURTHER RESULTS

There are several other configurations which yield products of binomial coefficients which are squares. For instance, if two hexagons H_1 and H_2 have a common entry, then the ten terms obtained by omitting the common entry have a product which is an integral square. Thus, one can build up a long serpentine configuration, or in fact build up snowflake curves.

Secondly, it should be noted in passing that all results above hold for generalized binomial coefficient arrays, in particular for the FIBONOMIAL COEFFICIENTS.