FIBONACCI, LUCAS, AND THE EGYPTIANS

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1. INTRODUCTION

One of the obvious distinctions between Egyptian mathematics and the mathematics of other cultures is its additive character of the dependent arithmetic. A typical example is recognized when we examine the algorithm used by the Egyptians in doing multiplication in comparison to other algorithms.

Multiplication (Egyptian Style) is done by a doubling-summing process similar to the one shown in the following example. Let us solve the following problem: 19 x 65. The Egyptians noted that the number 19 was equal to 1 + 2 + 16 (the sum of powers of two), hence, by the addition of appropriate multiples of 65 the Egyptians arrived at the desired result. We may arrange the problem in the following way:

	1*	*65	
doubling			doubling
	2*	*130	
doubling			doubling
	4	260	
doubling			doubling
	8	520	
doubling			doubling
	16*	*1040	
	19	1235	

Upon careful examination of the processes used in this algorithm, we find that there are two basic concepts that contribute to its efficiency. Namely, they are the concepts of distributivity and completeness. The latter conceived by Professor Verner E. Hoggatt, Jr. [1].

We can easily identify the role which is played by the distributive law in the algorithm, for example, in the preceding problem 65 x 19 = 65 (1 + 2 + 16). However, the contribution made by the concept of completeness is not self-evident. Let us turn to the definition of completeness before we examine its role in the Egyptian algorithm.

Definition. A sequence S of positive integers is said to be complete if and only if every element n, where n is an element of the positive integers can be represented as a sum of distinct elements of S.

The sequence used in the Egyptian method of multiplication the author shall describe as T, where $T_n = 2^n$ $(n \geq 0)$. In order to show that T is complete, we must first prove the following lemma.

Lemma 1. $T_0 + T_1 + T_2 + T_3 + \cdots + T_{n-1} = T_n - 1$.

Proof. We shall prove this lemma by mathematical induction. Here, we have

$$P(n) : T_0 + T_1 + T_2 + T_3 + \cdots + T_{n-1} = T_n - 1 .$$

Then $P(1) : T_0 = T_1 - 1$ is easily seen to be true since $1 = 2 - 1$. Thus, we have accomplished our inductive basis.

Now, suppose that

$$P(K) : T_0 + T_1 + T_2 + T_3 + \cdots + T_{k-1} = T_k - 1$$

is true (the inductive assumption), and we must then prove:

$$P(K + 1) : T_0 + T_1 + T_2 + T_3 + \cdots + T_k = T_{k+1} - 1.$$

By our inductive assumption, we know that

$$T_0 + T_1 + T_2 + T_3 + \cdots + T_{k-1} = T_k - 1 .$$

Hence, by substitution into $P(k + 1)$, we have that

$$T_k - 1 + T_k = T_{k+1} - 1.$$

It follows that

$$2T_k - 1 = T_{k+1} - 1 ,$$

hence, $2T_k = T_{k+1}$. Since $T_k = 2^k$, we have that $2T_k = T_{k+1}$. Therefore, we have shown that if $P(K)$ is true, then $P(K + 1)$ is true, and we have completed the inductive transition.

Employing Lemma 1, we may prove the following theorem.

Theorem 1. The sequence T, where $T_n = 2^n$ $(n \geq 0)$ is a complete sequence.

Proof. As an inductive basis, we know that

$$\begin{aligned} 1 &= 1 \\ 2 &= 2 \\ 3 &= 1 + 2 \\ 4 &= 4 \\ 5 &= 1 + 4 \\ 6 &= 2 + 4 \\ 7 &= 1 + 2 + 4, \quad \text{etc.} \end{aligned}$$

Hence, we must assume that there are representations for all the positive integers N:

$$1 < N < 2^{n+1} - 1.$$

Therefore, we must show that there are representations for all positive integers M:

$$2^{n+1} - 1 < M < 2^{n+2} - 1 .$$

By subtracting 2^{n+1} from the above inequality, we have that

$$-1 < M - 2^{n+1} < 2^{n+2} - 2^{n+1} - 1 .$$

Let $Q = M - 2^{n+1}$; hence, $-1 < Q < 2^{n+1} - 1$. This leads us to the conclusion that Q is representable as a sum of powers of 2 by our inductive assumption. And, from this, we can conclude that M is representable as a sum of powers of 2 since $M = Q + 2^{n+1}$ and

$$2^{n+1} - 1 = 1 + 2 + 2^2 + 2^3 + \cdots + 2^n .$$

Hence, we have completed our inductive transition.

2. FIBONACCI-EGYPTIAN METHOD

As we noted in the introduction, the necessary and sufficient conditions for the Egyptian algorithm to "work" are completeness and distributivity.

The author, upon reaching this conclusion, went in search of other sequences that would satisfy the above conditions. The first sequence examined proved to be fruitful. It was the Fibonacci sequence. It is obvious that the distributive law is satisfied, since we are working solely with positive integers; however, it is not so obvious that the Fibonacci sequence is complete. Let us then prove this fact.

As before, we must prove a lemma before proving the main theorem. It is the following:

Lemma 2.

$$F_{n+2} - 1 = F_1 + F_2 + F_3 + F_4 + \cdots + F_n .$$

Proof. We shall prove the lemma by mathematical induction.

$$P(n) : F_{n+2} - 1 = F_1 + F_2 + F_3 + F_4 + \cdots + F_n .$$

Then $P(1) : F_3 - 1 = F_1$ which is true, since $2 - 1 = 1$. Thus, we have accomplished our inductive basis. Now we must suppose that

$$P(K) : F_{k+2} - 1 = F_1 + F_2 + F_3 + F_4 + \cdots + F_k$$

is true (the inductive assumption), and we must then prove:

$$P(K + 1) : F_{k+3} - 1 = F_1 + F_2 + F_3 + F_4 + \cdots + F_{k+1} .$$

By the addition of F_{k+1} to both sides of the equation $P(K)$, we have

$$F_{k+2} + F_{k+1} - 1 = F_1 + F_2 + F_3 + \cdots + F_k + F_{k+1},$$

which lieads us to

$$F_{k+3} - 1 = F_1 + F_2 + F_3 + \cdots + F_{k+1}$$

by the recursion relation for Fibonacci numbers, namely

$$F_{n+3} = F_{n+2} + F_{n+1} .$$

Using this lemma, we may prove the following theorem.

Theorem 2. The Fibonacci numbers form a complete sequence.

Proof. The inductive proof will be considered in the following way. We observe that

$$\begin{aligned}
1 &= F_1 = F_2 \\
2 &= F_3 = F_2 + F_1 \\
3 &= F_4 = F_3 + F_2 \\
4 &= F_4 + F_2 = F_3 + F_2 + F_1, \quad \text{etc.}
\end{aligned}$$

We shall use this as our inductive basis. Next, we must assume that there are representations for all positive integers N, such that

$$1 < N < F_{n+2} - 1$$

is true. We must therefore show that there are representations for all positive integers M, such that

$$F_{n+2} - 1 < M < F_{n+3} - 1.$$

By subtracting an F_{n+2} from the above inequality, we have that

$$-1 < M - F_{n+2} < F_{n+3} - F_{n+2} - 1 .$$

Let $Q = M - F_{n+2}$; hence,

$$-1 < Q < F_{n+1} - 1 .$$

This leads us to the conclusion that Q is representable as a sum of Fibonacci numbers by our inductive assumption. And from this, we may conclude that M is representable as a sum of Fibonacci numbers, since

$$M = Q + F_{n+2}$$

and

$$F_{n+2} - 1 = F_1 + F_2 + F_3 + \cdots + F_n .$$

Hence, we have completed our inductive argument.

Let us examine the Fibonacci-Egyptian method for multiplication. For example, consider the problem 19 x 65. We note that

$$19 = 1 + 5 + 13 ,$$

all of which are Fibonacci numbers. Together with the Fibonacci recursion relation, and the following set-up, we may approach the problem in the following way:

$$\begin{array}{rr@{\qquad}rl}
 & 1* & *\ 65 & \\
+ & 2 & 130 & + \\
+ & 3 & 195 & + \\
+ & 5* & *325 & + \\
+ & 8 & 520 & + \\
+ & 13* & *845 & + \\
\hline
 & 19 & 1235 &
\end{array}$$

One may observe that in the preceding example, the entire Fibonacci sequence was not used. Upon examination, one will find that the first number of the sequence has been truncated. This does not, however, effect either the completeness of the sequence nor the distributivity. The author shall refer to the Fibonacci sequence with one element omitted as <u>the Deleted F Sequence</u>. Hence, let us prove the following theorem.

<u>Theorem 3.</u> The deleted F sequence, where $f_n = F_n$ $(n \geq 1)$ with arbitrary F_n not used, is complete.

Proof. From the previously proven theorem, it was noted that we may represent any positive integer n, where $1 \leq n \leq F_{n+1} - 1$ by using only the Fibonacci numbers F_1 through F_{n-1}, without using F_n. Hence, we shall consider F_n as the arbitrary Fibonacci number to be omitted. We may observe that F_{n+1} can represent itself. Since this is true, it is noted that we now have representations for $1 \leq n \leq 2F_{n+1} - 1$. Since we have increased our upper bound from what it was formerly, we may use this particular technique so that we may have representations for any positive integer without using F_n. For example, if $F_n = 1$, which is proposed to be the deleted number, then the sequence would remain complete.

Therefore, we have another method for multiplication which may be employed by those who have not mastered the traditional algorithm.

3. LUCAS-EGYPTIAN METHOD

Another sequence which proves fruitful in using our algorithm is the Lucas sequence. The Lucas sequence is composed of the numbers

$$(1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ \cdots)$$

and can be used effectively for the base sequence in an Egyptian multiplication problem. However, there is one acute difficulty in the consideration of this sequence for our algorithm; it does not have any representation for the positive integer 2. Therefore, something must be done to the sequence before we can apply it to our algorithm, since without a representation for the number 2 it is not complete.

The author chose to augment the sequence in the following way and define his Augmented Lucas Sequence as $A_n = L_{n-1}$, where $A_1 = 2$, $A_2 = 1$, $A_3 = 3$, and so on.

The reader will observe that this augmented sequence has a representation for 2 and also observe the recursion relation for the Lucas Sequence, namely $A_{n+1} = A_n + A_{n-1}$. Hence, we may use it for our base sequence in the Egyptian algorithm. The problem 18 x 54 may be set up in the following fashion.

	2	108	
+	1	54	+
+	3	162	+
+	4	216	+
+	7	378	+
+	11	594	+
+	18	972	+
	18	972	

The augmented Lucas sequence is complete and may be proved to be in a similar fashion to Theorem 2, by use of Lemma 3, which states

Lemma 3.

$$L_0 + L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 1.$$

Proof. Using an inductive proof, we have as our basis

$$P(1) : L_0 + L_1 = L_3 - 1$$

which is true, since $2 + 1 = 4 - 1$. Our inductive assumption is

$$P(K) : L_0 + L_1 + L_2 + L_3 + \cdots + L_k = L_{k+2} - 1\,.$$

We must then prove that

$$P(K + 1) : L_0 + L_1 + L_2 + L_3 + \cdots + L_{k+1} = L_{k+3} - 1$$

is true. This may be accomplished by adding a L_{k+1} to both sides of $P(K)$. Hence, we have that

$$L_0 + L_1 + L_2 + \cdots + L_k + L_{k+1} = L_{k+2} + L_{k+1} - 1,$$

which leads us to the fact that

$$L_0 + L_1 + L_2 + L_3 + L_4 + \cdots + L_{k+1} = L_{k+3} - 1\,.$$

Hence, our induction transition is complete.

Invoking this lemma, we may prove the following theorem.

Theorem 4. The augmented Lucas sequence is complete.

Proof. As our inductive basis, we have that

$$\begin{aligned} 1 &= L_1 \\ 2 &= L_0 \\ 3 &= L_2 \\ 4 &= L_3 \\ 5 &= L_3 + L_1, \quad \text{etc.} \end{aligned}$$

As our inductive assumption, we assume that for N, a positive integer, there are representations for N in terms of Lucas numbers so that

$$1 < N < L_{n+2} - 1 .$$

Hence, we must prove that for M, a positive integer, M is representable as a sum of Lucas numbers between the intervals of

$$L_{n+2} - 1 < M < L_{n+3} - 1 .$$

Using the same idea as described in the previously proven theorems, we shall subtract an L_{n+2} from the above inequality. Hence, we have that

$$-1 < M - L_{n+2} < L_{n+3} - L_{n+2} - 1 .$$

Let $Q = M - L_{n+2}$. Therefore,

$$-1 < Q < L_{n+3} - L_{n+2} - 1.$$

This leads us to the conclusion that

$$-1 < Q < L_{n+1} - 1 .$$

We may conclude that Q is representable as a sum of augmented Lucas numbers. And from this, we can conclude that M is representable as a sum of augmented Lucas numbers, since $M = Q + L_{n+2}$.

Other sequences may be investigated and tested for completeness; however, no others with starting values other than (1,1), (1,2), and (2,1) will be found which satisfy the generalized Fibonacci recursion relation. In general, other sequences that are complete will follow the following generalized recursion relation

$$G_n = \sum_{q=n-j}^{n-1} G_q \qquad j = (2, 3, 4, \cdots),$$

and where the starting values for the above sequences are taken from either the augmented Lucas sequence or the deleted F sequence. For example, let us examine the Tribonacci sequence, where three numbers are added. The generalized recursion relation would look like the following:

$$G_n = \sum_{q=n-3}^{n-1} G_q .$$

Hence, the sequence would be

$$(1, 2, 3, 6, 11, 20, \cdots).$$

In general j determines the number of terms to be added together and also the number of starting values to be taken from either the deleted F sequence or the augmented Lucas sequence.

The author at this point feels that it would be valuable for the reader to have a simple method for determining whether a sequence is or is not complete. It was observed and proven by John L. Brown, Jr. [2] that the necessary and sufficient conditions for a sequence to be complete is that the sequence satisfy the following general summation formula

$$A_{n+1} \leq 1 + \sum_{i=1}^{n} A_i$$

where $A_1 = 1$. Hence, we now have a convenient way in which to determine a sequence complete.

The material submitted in this paper is not completely theoretical and does have very definite practical application. The author used both the deleted F sequence and the augmented Lucas sequence in conjunction with the Egyptian method in a class of "slow learners." The results were phenomenal. Those students who could not multiply by traditional means were then given a method even they could handle. You see, all one needs to be proficient in the methods given above is an adequate understanding of simple addition. The author found that most slow learners could add correctly, however, they could not multiply. Therefore, this algorithm best fit the needs of those students.

The concepts mentioned throughout the paper may also be used in advanced mathematics classes. Hence, as one can see, the utility of these topics and their applications is boundless.

It is the author's intent that the reader search for other complete sequences and establish those concepts revealed in this paper, so that he may transfer the concepts to others and hence, give many an algorithm for multiplication which they may not already have.

The author would also like the reader to be aware of the fact that it is sometimes advantageous to use one complete sequence over another. For example, it is better to use the Lucas sequence when multiplying the numbers 18 x 432, than it is to use the Fibonacci sequence or the powers of two sequence, since 18 is an element of the Lucas sequence. Therefore, this was the primary reason the author went in search of other complete sequences.

The author hopes that the methods for multiplication developed in this paper will be tried, and hopes that the success of those using them will be as rich as his own.

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