ASYMPTOTIC EUCLIDEAN TYPE CONSTRUCTIONS WITHOUT EUCLIDEAN TOOLS [1]

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INTRODUCTION

> "..., Gauss made the remarkable discovery that those, and only those, regular polygons having a prime number of sides p can be constructed with straight edge and compasses if and only if p is of the form $2^{2^n} + 1$. Now the ancient Greeks had found how to construct with straight edge and compasses regular polygons of 3, 4, 5, 6, 8, 10 and 15 sides. If in the formula $p = 2^{2^n} + 1$ we set n = 0 and 1, we obtain the primes 3 and 5 respectively — cases already known to the Greeks. For n = 2, we find p = 17, which is a prime number. Therefore Gauss showed that a regular polygon of 17 sides is constructible with straight edge and compasses, which was unknown to the Greeks. Gauss was vastly proud of this discovery, and he said that it induced him to choose mathematics instead of philology as his life work." [2]

This quote from Howard W. Eves' recent two-volume set, In Mathematical Circles, suggests that the construction of regular polygons having a prime number of sides is not easy, even when possible, with a straight edge and compass. Note that Gauss showed it is impossible to construct with a ruler and compass the regular seven-sided polygon. Furthermore, one method for showing that a general angle θ cannot be trisected with Euclidean tools involves showing that it is impossible to trisect the angle whose

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[2]Howard W. Eves, In Mathematical Circles, Quadrants III and IV, Prindle, Weber and Schmidt, Inc., Boston, 1969, p. 113.

measure is $\pi/3$ — hence, the nine-sided regular polygon is not constructible with a ruler and compass either.*

The first part of this article deals with a way to approximate, by folding a paper strip, any regular polygon whose number of sides is of the form $2^n \pm 1$, for some natural number n. Note that when $n = 3$, the expression $2^n \pm 1$ yields 7 and 9.

A modification of the iterative folding sequences used on paper strips is presented. It suggests a method for approximating an angle having measure $\theta/(2^n + 1)$, where n is any natural number and θ is any given angle whose measure is between 0 and π Particularly interesting is the case when $n = 1$, which produces a trisection approximation process.

Finally, as an illustration, instructions are given describing how paper strips may be used to construct models of regular convex dodecahedra. The constructions suggest, as will be seen, that a "parallel strip" classification of certain polyhedra might provide an interesting point of view from which to study their properties.

FOLDING SEQUENCES INVOLVING ONE ITERATIVE EQUATION

As an elementary example, take a roll of ordinary adding machine tape and make a fold on any straight line, t_0, near the end of the tape so that t_0 crosses one of the parallel edges of the tape at a point, A_0. Fold again through A_0 to bisect one of the angles formed by t_0 and an edge of the tape. Do this so that the newly created transversal, t_1, goes towards the roll of paper. One endpoint of t_1 is A_0, the other endpoint is named "A_1." Now fold the tape through A_1, bisecting the obtuse angle created by t_1 and the edge of the tape. This fold yields yet another transversal, t_2, whose endpoints are A_1, A_2. To continue this folding process always bisect, by folding through A_n, the obtuse angle, having sides t_n and an edge of the tape; thereby obtaining a new transversal, t_{n+1}, having endpoints A_n, A_{n+1} (for $n = 1, 2, 3, \cdots$). The acute angle formed by t_n and an edge of the tape is denoted x_{n-1}.

*Howard W. Eves, An Introduction to the History of Mathematics, Rinehart and Company, Inc., New York, 1953, pp. 96-98, p. 107.

For the most accurate results, both in this case and all other examples which follow, fold the tape so that whenever transversals are formed, the tape remains folded on these creases and the next fold always occurs on the portion of the paper strip which comes from the top of the existing configuration. Thus, the triangles which are formed will either stack up or form a zig-zag type pattern in the folding plane, but the configuration formed will never need to be turned over during the folding process. One quickly discovers, however, that certain rotations of the configuration in the folding plane facilitates the folding process. Figure 1 illustrates one case of how the unfolded tape appears after the folding process has taken place.

When the above folding process is accurately carried out, an accordian-like stack of triangles results. And, it soon becomes visually apparent that successive triangles are getting more and more alike — consequently, the measure of x_n must approach $\pi/3$ as n gets large.

For skeptics, the proof can be ascertained. First, note that since the edges of the tape are parallel, the measures of successive acute angles always satisfy the equation

$$2x_n + x_{n-1} = \pi \quad ,$$

where $n = 1, 2, 3, \cdots$. Successive computations of x_1, x_2, x_3, etc., yields

$$x_n = \frac{\pi}{2}\,[1 + (-1/2)^1 + (-1/2)^2 + \cdots + (-1/2)^{n-1}] + (-1/2)^n x_0 \quad ,$$

which can be verified by mathematical induction. Then, using the formula for the sum of a geometric sequence, it follows that

$$x_n = \frac{\pi}{3}\,[1 - (-1/2)^n] + (-1/2)^n x_0 \ .$$

Consequently,

$$\lim_{n\to\infty} x_n = \pi/3 \ .$$

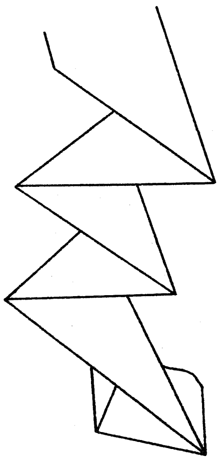

Figure 2(a)

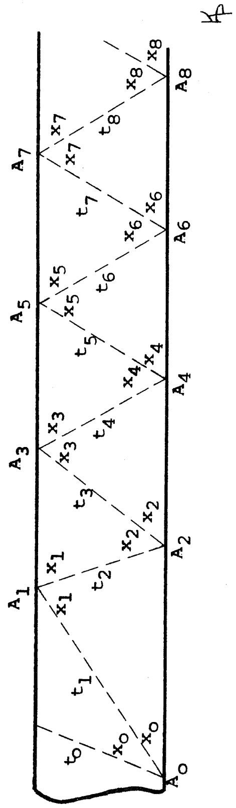

Figure 1

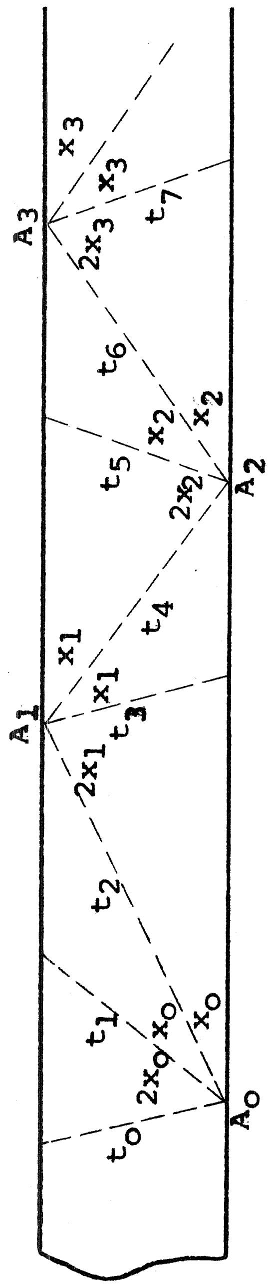

Figure 2(b)

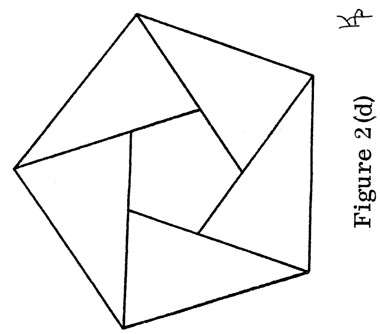

Figure 2(d)

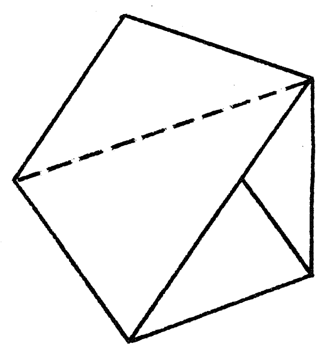

Figure 2(c)

Notice that the difference, in radians, between x_n (which is formed by the $(n+2)^{nd}$ fold of the tape) and $\pi/3$ is

$$(-1/2)^n [x_0 - \pi/3] .$$

This means each new accurate fold on the tape produces an angle whose measure is twice as close to $\pi/3$ as its predecessor. In fact, the maximum value for the actual error (which occurs when x_0 approaches zero) indicates that one can always expect an approximation of $\pi/3$ with accuracy better than one minute after 14 folds. But, as one is not likely to choose x_0 close to zero, this degree of accuracy will occur, in most cases, when $n < 14$.

It turns out to be practical, in the paper tape construction of models, to have the following:

Visual Criterion. When the consecutive longest transversals formed on a tape by an iterative folding process, appear to be of the same length, then the tape is called usable.

Suppose the length of successive transversals obtained from some iterative folding process approaches some fixed value, in the limit sense. Then there must exist some number $k \neq 0$ such that consecutive acute angles formed by those transversals and an edge of the tape converge to an angle having measure π/k.

Definition. On a usable tape, whose successive smallest acute angles converge to π/k; when the portion not satisfying the Visual Criterion is cut off, the remaining tape is denoted "$T(\pi/k)$."

Accordingly, the usable tape produced in the above example is denoted "$T(\pi/3)$" and called a "pi thirds tape."

The method of obtaining $T(\pi/k)$ implies that there will always be some natural number, p, such that the transversals t_n, where $n < p$, will not appear on that tape. But, it is not necessary to identify p. Thus, in describing constructions, reference to a transversal t_n on $T(\pi/k)$ will mean any transversal on $T(\pi/k)$. However, once t_n has been identified for use in a particular construction, then t_{n+q} (where q is any natural number) will mean the q^{th} transversal following t_n.

Since $T(\pi/3)$ contains approximations of equilateral triangles, it may be used to construct models of hexagons and deltahedra. As an example, cut $T(\pi/3)$ on t_n and t_{n+10}, then fold the ten triangle strip on t_{n+3} and t_{n+6}. Now, because straight lines are easier to fold than to cut, the t_{n+10} end of the tape is wrapped around t_n when the tape is folded on t_{n+9} to complete the model of a hexagon. Note that the definitive edges do not include either of the cut edges t_n, t_{n+10}.

The above folding process generalizes in the following way.

<u>Theorem 1.</u> If

(1) n is some fixed natural number.

(2) A paper tape of width w is folded on some transversal, t_0, which crosses one of the parallel edges of the tape at A_0.

(3) One angle formed by t_0 and an edge of the tape is then divided into 2^n parts, by folding through A_0; creating, in order, the new set of transversals, $t_1, t_2, t_3, \cdots, t_n$, where $t_1 < t_2 < t_3 < \cdots < t_n$. The measure of the acute angle formed by t_n and the edge of the tape is denoted x_0. The endpoint of t_n which lies on the opposite edge of the tape from A_0 is called A_1.

(4) In general, folds are made so as to divide into 2^n parts the obtuse angle having vertex A_k and an interior with no transversals. The new transversals, $t_{kn+1}, t_{kn+2}, \cdots, t_{kn+n}$, are such that $t_{kn+1} < t_{kn+2} < \cdots < t_{kn+n}$. The endpoint of t_{kn+n}, called A_{k+1}, lies on the opposite edge of the tape from A_k (for $k = 1, 2, 3, \cdots$). The measure of the acute angle formed between t_{kn+n} and an edge of the tape is denoted x_k.

Then $\lim_{k\to\infty} x_k = \pi/(2^n + 1)$ and consequently, this folding process produces $T(\pi/(2^n + 1))$.

<u>Proof.</u> From the description of the folding process, it follows that the measures of successive acute angles satisfy the equation

$$2^n x_k + x_{k-1} = \pi \quad , \tag{1}$$

where $k = 1, 2, 3, \cdots$. Then, using mathematical induction, it can be shown that

$$x_k = \frac{\pi}{2^n + 1}\left[1 - (-1/2^n)^k\right] + (-1/2^n)^k x_0 \ ,$$

for $k = 1, 2, 3, \cdots$. But, since $\left|-1/2^n\right| < 1$, it follows immediately that

$$\lim_{k \to \infty} x = \pi/(2^n + 1) \ .$$

The theorem is surprisingly fruitful. For example, Figure 2(a) illustrates how the folded tape appears just after the folding process has taken place with $n = 2$. Figure 2(b) shows how this same tape appears when it is unfolded. This folding process produces the usable tape, $T(\pi/5)$. If $T(\pi/5)$ is cut on t_{2n} and t_{2n+6}, and folded on t_{2n+1}, t_{2n+3}, t_{2n+5}, a model of the regular pentagon shown in Figure 2(c) is formed. The sides of this pentagon approximate $w/\sin(2\pi/5)$. But, a regular pentagon whose sides approximate $w/\sin(\pi/5)$ may also be formed from $T(\pi/5)$. To see this, cut $T(\pi/5)$ on t_{2n+1} and t_{2n+13}; then fold in a winding fashion on the transversals t_{2n+2}, t_{2n+4}, t_{2n+6}, t_{2n+8}, t_{2n+10}, t_{2n+12}. The result, a model of a regular pentagon with a pentagonal hole in the center, is shown in Figure 2(d).

As another example, consider the results of the theorem when $n = 3$. Figure 3(a) shows how the beginning of the tape which produces $T(\pi/9)$ might appear. Once $T(\pi/9)$ has been obtained, it may be used to construct models of regular 9-gons whose sides approximate either $w/\sin(\pi/9)$, $w/\sin(2\pi/9)$, or $w/\sin(4\pi/9)$. This is done by folding $T(\pi/9)$ on consecutive transversals whose labels are equal to $0 \pmod 3$, $2 \pmod 3$, and $1 \pmod 3$, respectively. Figure 3(b) illustrates the regular 9-gon which is formed by folding on t_{3n+1}, t_{3n+4}, t_{3n+7}, $\cdots$, t_{3n+28}; and whose sides approximate $w/\sin(4\pi/9)$.

In general, $T(\pi/(2^n + 1))$ will produce models of n non-congruent regular $(2^n + 1)$-gons whose sides approximate $w/\sin(2^k\pi/(2^n + 1))$, $k = 1, 2, \cdots, (n - 1)$. The actual construction involves folding $T(\pi/(2^n + 1))$ on successive transversals whose labels are equal to $0 \pmod n$, $(n - 1) \pmod n$, $(n - 2) \pmod n$, $\cdots$, $1 \pmod n$, respectively.

A BONUS

Suppose the folding process described in the theorem takes place on a piece of paper whose straight edges are <u>not</u> parallel. Thus, suppose angle

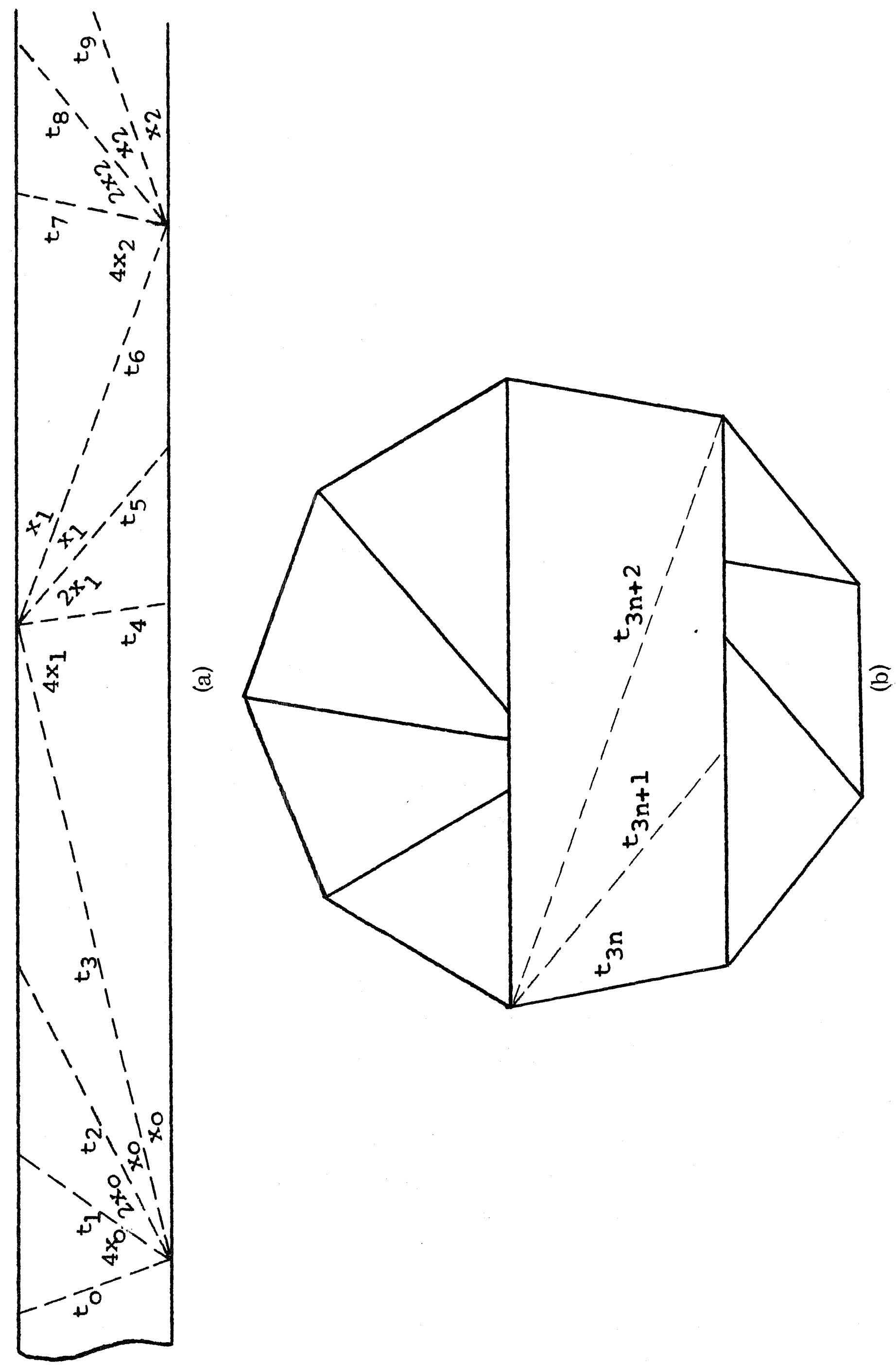
t0
4x0
t1
2x0
x0
x0
t2
t3
4x1
t4
2x1
x1
x1
t5
t6
4x2
t7
2x2
x2
x2
t8
t9
(a)
t3n
t3n+1
t3n+2
(b)

Figure 3

ABC, having measure θ (between 0 and π), and supplementary angle ABD, occur so that DBC lies on the edge of a piece of paper. Then the paper is cut along the line AB (see Figure 4). A point, A_0, is selected between D and B and the paper is folded, through A_0 on some line, t_0, which is not parallel to AB. The transversals t_k (where $k = 1, 2, \cdots$) are formed by folding so that t_1 bisects the angle formed by t_0 and A_0B, determining a point, A_1, on the line containing AB. And, in general, t_k bisects the angle $A_{k-2}A_{k-1}B$, determining a point A_k on the line containing $A_{k-2}B$ (when $k \geq 2$). The measure of the angle A_1A_0B is denoted x_0 and half the measure of angle $A_{k-2}A_{k-1}B$ is denoted x_{k-1} for $k \geq 2$.

Then, since the sum of the measures of the interior angles in any triangle is always equal to π it follows that

$$2x_k + x_{k-1} + (\pi - \theta) = \pi \quad ,$$

when $k = 1, 2, 3, \cdots$. Thus,

$$2x_k + x_{k-1} = \theta \ , \tag{2}$$

when $k = 1, 2, 3, \cdots$.

But this is similar to Equation 1, where $n = 1$. In fact, a review of the proof for Theorem 1 reveals that it would not have been any more difficult if "π" were replaced with "θ" and that Equation 2 would lead to the result

$$\lim_{k \to \infty} x_k = \theta/3 \ .$$

Thus the method illustrated in Figure 4 really represents a <u>trisection approximation</u> method for angles whose measure is between 0 and π. As a practical matter it is not, in this case, possible to fold accurately indefinitely, as was the case with parallel lines. Nevertheless, the method is effective — especially when judicious choices of A_0 and t_0 are made — i.e., choose A_0 as far away from B as the paper will allow and make a visual guess when folding t_0 so that when x_0 is formed, it will be as close to $\theta/3$ as possible.

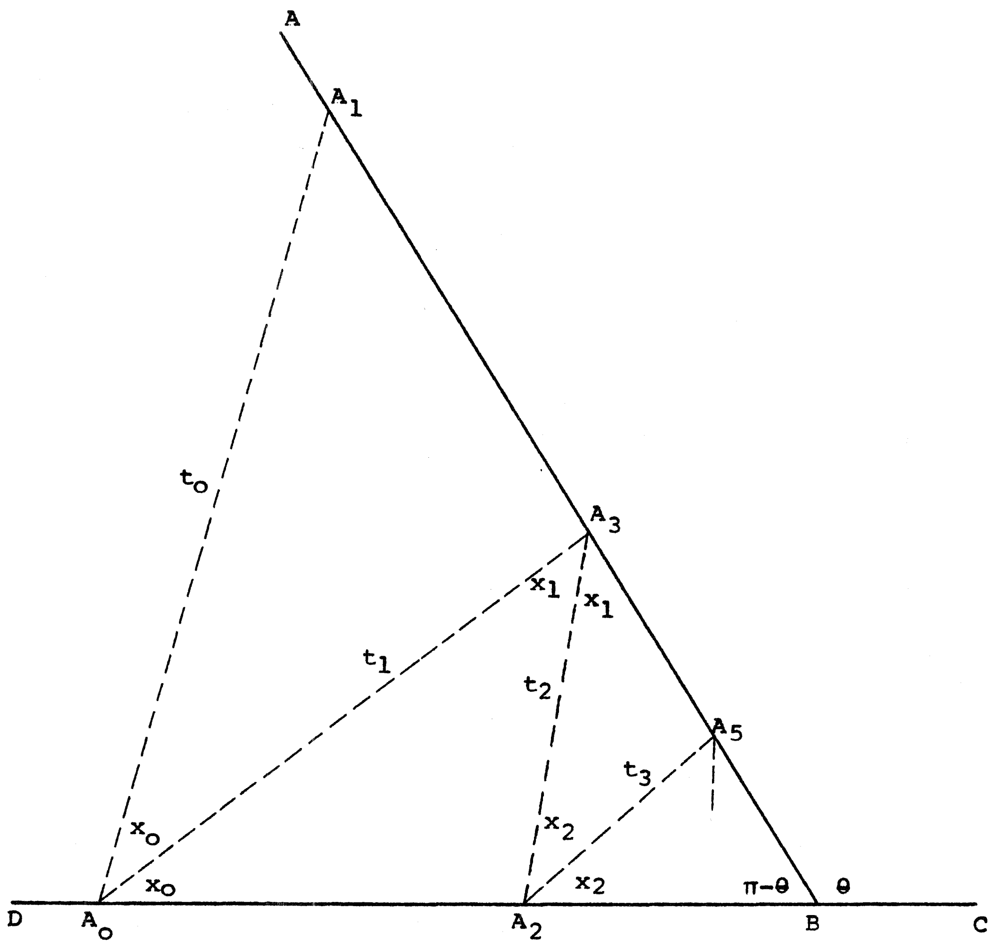

Figure 4

OTHER FOLDING SEQUENCES

The folding sequences considered thus far have involved just one iterative equation. But, as the next theorem shows, other folding sequences do exist.

Theorem 2. If

(1) n is some fixed natural number greater than 1.

(2) A paper tape, of width w, is folded on a transversal, t_0, which crosses an edge of the tape at some point, A_0.

(3) The angle formed by t_0 and the edge of the tape having vertex A_0 is divided, by folding, into 2^n parts producing transversals t_1,

$t_2, t_3, \cdots, t_n$ so that $t_1 < t_2 < t_3 < \cdots < t_n$; and t_n has endpoints A_0, A_1. The measure of the acute angle which t_n makes with the edge of the tape is denoted x_0.

(4) The obtuse angle at A_1 is bisected, creating a new transversal t_{n+1}. It has endpoints A_1, A_2 and forms an acute angle with an edge of the tape, denoted x_1.

(5) In general, either (i) the obtuse angle at A_k is divided into 2^n parts, when k is even, so that each new transversal is longer than its predecessor and the last transversal folded creates the point A_{k+1} on the opposite edge of the tape from A_k; or (ii) the obtuse angle at A_k is bisected, when k is odd. In either case, the measure of the acute angle between the transversal joining A_k, A_{k+1} and an edge of the tape is denoted x_k.

Then $\lim_{k \to \infty} x_{2k} = \pi/(2^{n+1} - 1)$ and, consequently, the folding sequence produces $T(\pi/(2^{n+1} - 1))$.

Proof. By the description of the folding process, it follows that the measures of consecutive acute angles satisfy

$$\left.\begin{aligned} 2x_{2k-1} + x_{2k-2} &= \pi \\ 2^n x_{2k} + x_{2k-1} &= \pi \end{aligned}\right\} \text{ for } k = 1, 2, 3, \cdots . \tag{3}$$

Solving for x_{2k-1} in the first iterative equation, then for x_{2k} in the second yields

$$x_{2k} = (\pi + x_{2k-2})/2^{n+1} .$$

It can then be shown, by mathematical induction, that

$$x_{2k} = \frac{\pi}{2^{n+1} - 1} \left[1 - (1/2^{n+1})^k\right] + (1/2^{n+1})^k x_0 ,$$

for $k = 1, 2, 3, \cdots$. Thus

$$\lim_{k \to \infty} x_{2k} = \pi/(2^{n+1} - 1) .$$

In general, if $T(\pi/(2^{n+1} - 1))$ is folded on all t_{k-1}, t_{k-2}, where $k = 0 \pmod{n+1}$, a regular $(2^{n+1} - 1)$-gon will be formed.

As an example, suppose $n = 2$ in Theorem 2. Figure 5(a) illustrates how the beginning of this tape, which produces $T(\pi/7)$, might appear.

If $T(\pi/7)$ is folded, in a winding fashion, on all t_{k-1}, t_{k-2}, where $k = 0 \pmod 3$, the model formed is a regular seven-sided polygon (Figure 5(b)), whose sides approximate $w/\sin(\pi/7)$.

Likewise, if $T(\pi/7)$ is folded on all t_k where $k \neq 1 \pmod 3$, the result is a seven-sided polygon whose sides approximate $w/\sin(3\pi/7)$. If this is done so that the folds on t_k, when $k = 0 \pmod 3$, wrap the tape around the polygon being formed; then the result appears as shown in Figure 5(c).

Note, however, that as illustrated in Figure 5(d), if $T(\pi/7)$ is folded on all t_k where $k \neq 2 \pmod 3$, a regular seven-sided <u>star</u> polygon is formed whose sides approximate $w/\sin(\pi/7)$. It can be shown that the shortest distance between consecutive vertices approximates $w/\sin(2\pi/7)$.

CONSTRUCTING DODECAHEDRA WITH $T(\pi/5)$

When cash register tape (which is more porous than adding machine tape) is used with white glue, surprisingly sturdy models of polyhedra may be made.

To construct a dodecahedron, for example, fold the cash register tape to obtain $T(\pi/5)$ containing at least 90 usable triangles. Cut $T(\pi/5)$ on t_{2n} and t_{2n+56}, then fold the resulting strip, glueing the overlapping portions in position as shown in Figure 6. Label the edges of the pentagons as shown. The polyhedron is completed by first forming a ring-like figure and glueing one of the shaded parallelograms on top of the other. Then join the remaining 18 pairs of edges so that edges labeled with like numbers correspond with each other. Tabs for joining the edges may be conveniently obtained by cutting on nineteen successive long transversals of $T(\pi/5)$.

If the tabs are labeled so that when they are glued in place it preserves the numbers shown on each of the edges, it is then possible, upon completion of the dodecahedron, to observe that

<u>The dodecahedron, formed from $T(\pi/5)$ of width w, and whose edge approximates $w/\sin(2\pi/5)$ may be constructed with no fewer than six</u>

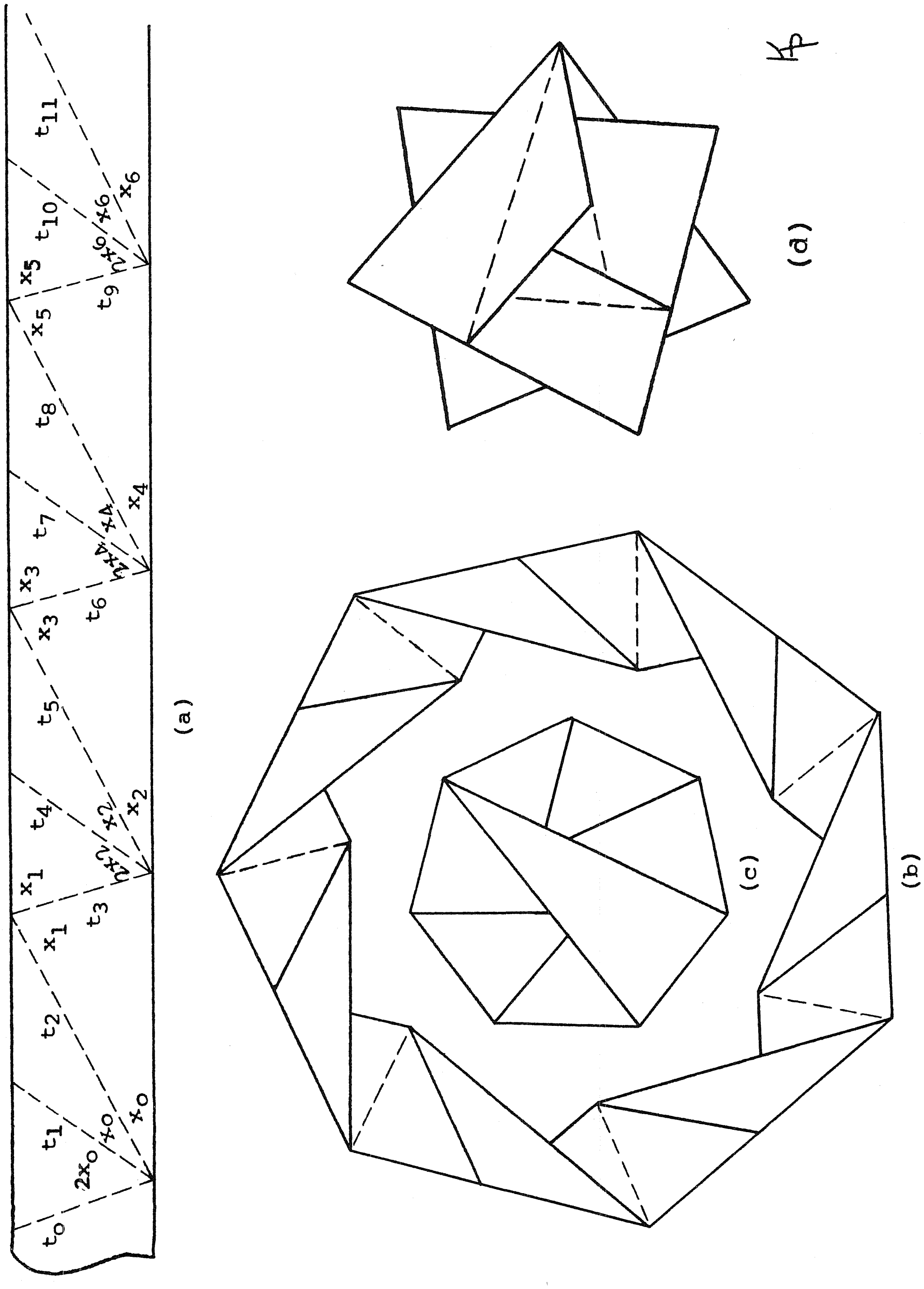

Figure 5

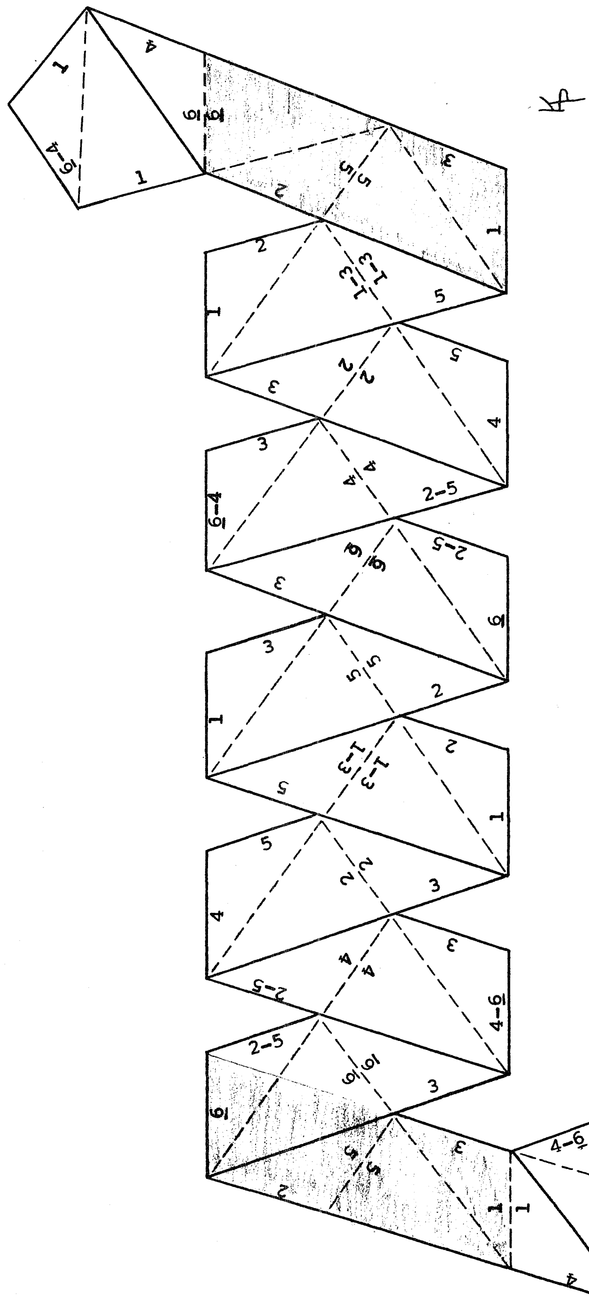

Figure 6

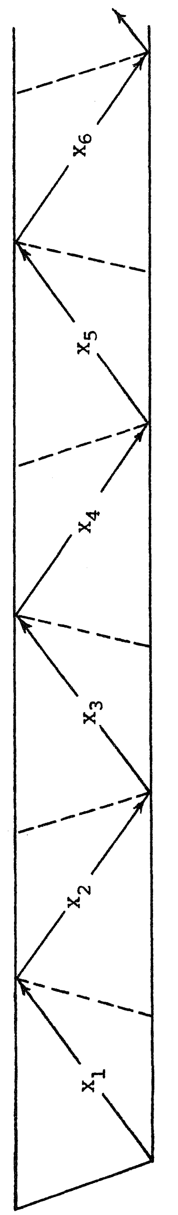

Figure 7

bands, each of which contains 12 consecutive triangles from $T(\pi/5)$. (In practice, an extra triangle would be required on some bands — but, since it serves only as a tab, it is not counted.)

To see that this is true, take a strip of $T(\pi/5)$ which contains 12 triangles and observe that it is possible to position it on the completed dodecahedron so that its short transversals all coincide with edges whose label includes the symbol "1." But, it may also be positioned in five other ways so that its short transversals coincide with the edges each of whose labels include the symbols "2," "3," "4," "5," and "6," respectively. Because the label on every edge contains at least one number, six bands are sufficient for this particular construction of the dodecahedron. Note that if any one number were removed from the labels on this dodecahedron, there would be some edges with no label. Therefore, at least six bands are necessary for the construction of this dodecahedron.

This model may be used to show that if a dodecahedron were constructed from six bands, each containing 12 consecutive triangles from $T(\pi/5)$, there would be six edges crossed by exactly two bands and those edges would be oriented so that (a) their midpoints are the vertices of an inscribed octahedron; (b) the collection of pentagonal diagonals parallel to those six edges form the edges of an inscribed cube; and, since alternative vertices of a cube define vertices of a tetrahedron, (c) the vertices of two distinct inscribed tetrahedra may be identified on this model.

A second, somewhat different, dodecahedron may be constructed using $T(\pi/5)$. This model is particularly easy to make from gummed tape. Cash register tape and white glue produce a better looking model but, having one side gummed makes the description of the construction easier. Accordingly, the following instructions are given for gummed tape.

First, cut from $T(\pi/5)$ six strips of 22 triangles each. The first portion of a typical strip is shown, with the gummed side down, in Figure 7. Label the ungummed side of each of the strips by replacing the letter "X" shown in Figure 7 with the letters "A," "B," "C," "D," "E," "F," on the first, second, third, fourth, fifth, and sixth strips, respectively. As an example, the first strip, called "strip A," will have its eleven long transversals labeled "A_1," "A_2," $\cdots$, "A_{11}," consecutively, and all transversals will be labeled with an arrow which points to the endpoint of the next long transversal.

The following notational device is convenient: If X, Y, Z represent members of $\{A, B, C, D, E, F\}$ and if m, n, k are natural numbers, then "$X_m \longrightarrow Y_n$" means that: the gummed side of strip X is glued onto the ungummed side of strip Y so that the transversal marked "$-X_m\rightarrow$" coincides with the transversal marked "$-Y_n\rightarrow$" and the arrows point in the <u>same</u> direction.

"$X_m \longleftrightarrow Y_n$" means that: the gummed side of strip X is glued onto the ungummed side of strip Y so that the transversal marked "$-X_m\rightarrow$" coincides with the transversal marked "$-Y_n\rightarrow$" and the arrows point in <u>opposite</u> directions.

"$X_m \longrightarrow Y_n \longrightarrow Z_k$" means that: $X_m \longrightarrow Y_n$ and $Y_n \longrightarrow Z_k$.

Using this notational device, the dodecahedron is assembled as follows:

I.		II.	
	$E_7 \rightarrow A_6$		$A_5 \leftrightarrow D_8$
	$D_7 \rightarrow E_6$		$B_5 \leftrightarrow E_8$
	$C_7 \rightarrow D_6$		$C_5 \leftrightarrow A_8$
	$B_7 \rightarrow C_6$		$D_5 \leftrightarrow B_8$
	$A_7 \rightarrow B_6$		$E_5 \leftrightarrow C_8$

III. The F strip may now be woven in and out so that

$$D_9 \rightarrow F_2$$
$$F_3 \rightarrow B_4$$
$$E_9 \rightarrow F_4$$
$$F_5 \rightarrow C_4$$
$$A_9 \rightarrow F_6$$
$$F_7 \rightarrow D_4$$
$$B_9 \rightarrow F_8$$
$$F_9 \rightarrow E_4$$
$$C_9 \rightarrow F_{10}$$
$$F_{11} \rightarrow F_1 \rightarrow A_4$$

IV.	$A_3 \longleftrightarrow C_{10}$	V.	$A_{11} \longrightarrow A_1 \longrightarrow E_2$
	$B_3 \longleftrightarrow D_{10}$		$B_{11} \longrightarrow B_1 \longrightarrow A_2$
	$C_3 \longleftrightarrow E_{10}$		$C_{11} \longrightarrow C_1 \longrightarrow B_2$
	$D_3 \longleftrightarrow A_{10}$		$D_{11} \longrightarrow D_1 \longrightarrow C_2$
	$E_3 \longleftrightarrow B_{10}$		$E_{11} \longrightarrow E_1 \longrightarrow D_2$

This dodecahedron is also formed from exactly six bands, but each band contains 20 triangles (not counting the overlapping tabs) from $T(\pi/5)$. Comparing the two completed polyhedra, one will note many similarities and differences. The first and most obvious difference is that the one has some holes in it and that it appears to be "woven together." A most effective model of the second dodecahedron may be made if six different colored strips are used in its construction. In fact, it is not even necessary to use glue, for one can hold the various strips together as indicated by the instructions, with 30 paper clips. Then, when the dodecahedron is finished, all of the paper clips, except those six which hold three thicknesses of tape together, may be removed.

If the places where bands overlap themselves are discounted, all of the edges of the second dodecahedron are crossed by exactly two bands. If one imagines the arrows on this dodecahedron to be roads on which travel is permitted only in the direction of the arrows, it can be seen that, if one leaves the pentagonal cycle $A_{11}B_{11}C_{11}D_{11}E_{11}$, all roads lead to the cycle

$$D_9 F_3 E_9 F_5 A_9 F_7 B_9 F_9 C_9 F_{11}$$

and, leaving that cycle, all roads lead to the cycle $A_7B_7C_7D_7E_7$, from which there is no escape.

Many other polyhedra may be constructed with paper strips. If the reader wishes to try devising some paper tape constructions for other polyhedra, the following references may be useful.

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SPECIAL ADVANCED PROBLEM

H-182S Proposed by Paul Erdős, University of Colorado, Boulder, Colorado.

Prove that there is a sequence of integers $n_1 < n_2 < \cdots$ satisfying

$$\frac{\sigma(n_k)}{n_k} \to \infty \quad \text{and} \quad \frac{\sigma(\sigma(n_k))}{\sigma(n_k)} \to 1 ,$$

where

$$(n) = \sum_{d|n} d$$

(the sum of the integer divisors of n.)

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