KAPREKAR'S ROUTINE WITH TWO-DIGIT INTEGERS

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Kaprekar's routine consists of rearranging the digits (not all alike) of an integer, N_0, to form the largest and the smallest possible integers, finding their difference, N_1, and repeating the operation on N_1 and on the subsequent differences until a terminal situation is reached. He found [1] that when the routine is applied to any four-digit (not all alike) integer in the decimal system, the self-producing 6174 is eventually reached. The routine has been expanded to other number bases [2], [3], and to three-digit [4] and five-digit [5] integers.

When applied to two-digit integers, an integer and its reverse are involved, the smaller being subtracted from the larger in each step of the routine. In the system with base r, all differences are multiples of $r - 1$. Each step may be called a reversal-subtraction-operation (<u>RSO</u>).

There are three possible terminal situations which may result when the routine is applied to a two-digit integer, namely:

A. If at any step of the repetitious routine, an integer with two like digits is produced, its N_1 will be 00.

B. A <u>self-producing integer</u> is formed. That is, the integer is reproduced when subjected to an RSO. For example: 37 in the scale of eleven, where $73 - 37 = 37$.

C. A regenerative loop is formed, in which an RSO on one member produces the next member. For example: in the scale of nine, $53 - 35 = 17$, $71 - 17 = 53$, and so on.

In each of these categories, if all two-digit integers in a particular system lead to the same result, it is said to be <u>unanimous</u>.

The two-digit ordered integers ab and $\overline{a+k}\,\overline{b+k}$ have the same N_1. Hence, to investigate the entire field in the scale of r, it will be sufficient to examine only the $r - 1$ integers with the form $\overline{r-1}\,b$, where $b < r - 1$. Each of these is the representative (<u>rep</u>) of all two-digit integers from which an RSO produces the same N_1. Any two-digit integer can be converted into its rep by addition of an appropriate multiple of 11. Thus, $52 + 44 = 96$ in the decimal system, so 52 - 25 = 27 = 96 - 69. All integers with the same rep have the same value of $a - b$.

THE EXAMINATION PROCEDURE

An RSO is performed on each of the reps. Each difference, N_1, is converted into its own rep. These results are assembled into flow charts such as those given below. All N_1's with the same rep are placed below a common subtraction line. Their common rep is placed below them on the left. The category letter of each terminal situation is placed below it.

Thus, only $r - 1$ RSO's are necessary to examine the entire field in the base r. The number of steps necessary to convert any integer into the terminal result can be read directly from the chart after locating its rep. In the scale of five, three RSO's convert 12 (which has the rep 43) into the self-producing 13.

In the charts for bases eleven and twelve, the symbols X and E stand for the digits ten and eleven, respectively.

OPERATIONAL FLOW CHARTS

Base Two	Base Three	Base Four		Base Five	
10	21	30	31	43	41
01	12	03	13	34	14
01	02	21	12	04	22
B					A
	20	32		40	
	02	23		04	
	11	03		31	
	A	C			
				42	
				24	
				13	
				B	

Base Six

50	53	
05	35	
41	14	
52		51
25		15
23		32
54		
45		
05		
C		

Base Seven

65	63	61
56	36	16
06	24	42
60	64	
06	46	
51	15	
62		
26		
33		
A		

Base Eight

70	75		71
07	57		17
61	16		52
72		73	74
27		37	47
43		34	25
			B
76			
67			
07			
C			

Base Nine

85	81		83
58	18		38
26	62		44
			A
84		87	
48		78	
35		08	
86		80	
68		08	
17		71	
82			
28			
53			
C			

Base Ten

90	97	
09	79	
81	18	
92		95
29		59
63		36
96	91	
69	19	
27	72	
94		93
49		39
45		54
98		
89		
09		
C		

Base Eleven

X9	X5	X3	X7	X1
9X	5X	3X	7X	1X
0X	46	64	28	82
X0	X8		X4	
0X	8X		4X	
91	19		55	
			A	
X2				
2X				
73				
X6				
6X				
37				
B				

Base Twelve

E0	E9	
0E	9E	
X1	1X	
E2		E7
2E		7E
83		38
E6	E3	
6E	3E	
47	74	
E8		E1
8E		1E
29		92
E4	E5	
4E	5E	
65	56	
EX		
XE		
0E		
C		

SUMMARY AND GENERALIZATIONS

1. Every system with an odd base has a sequence leading to 00, since

$$\overline{r-1}\;\overline{(r-3)/2} - \overline{(r-3)/2}\;\overline{r-1} = \overline{(r-1)/2}\;\overline{(r-1)/2}\,.$$

In bases three and seven, 00 is unanimous.

2. If a self-producing integer, kx with $k < x$, exists in a system with base r, then

$$(rx + k) - (rk + x) = rk + x$$

whereupon

$$r = (2x - k)/(x - 2k) = 2 + 3k/(x - 2k) .$$

Then, since $x < r$, self-producing integers, which will have the form $k\ \overline{2k + 1}$, exist in and only in systems with bases of the form $3k + 2$.

Such bases are two (in which 01 is unanimous), five, eight, and eleven.

3. Both $\overline{r - 1}\ c$ and $\overline{r - 1}\ \overline{r - 3 - c}$ have N_1's which are the reverse of each other, since

$$\overline{r - 1}\ c - c\ \overline{r - 1} = \overline{r - 2 - c}\ \overline{c + 1}$$

and

$$\overline{r - 1}\ \overline{r - 3 - c} - \overline{r - 3 - c}\ \overline{r - 1} = \overline{c + 1}\ \overline{r - 2 - c} .$$

Hence, the N_1's have the same rep.

4. If r is even and <u>not</u> of the form $3k + 2$, the result of application of RSO's to the reps in that system is a unanimous regenerative loop of $r/2$ elements.

REFERENCES

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2. J. H. Jordan, "Self-Producing Sequences of Digits," <u>American Mathematical Monthly</u>, 71 (January, 1964), pp. 61-64.
3. Charles W. Trigg, "Predictive Indices for Kaprekar's Routine," <u>Journal of Recreational Mathematics</u>, scheduled for publication in October, 1970.
4. Charles W. Trigg, "All Integers Lead to ···," submitted to <u>The Arithmetic Teacher.</u>
5. Charles W. Trigg, "Kaprekar's Routine with Five-Digit Integers," submitted to <u>Mathematics Magazine.</u>