TRIANGLES DE FIBONACCI

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Consider a triangle whose sides have lengths represented by Fibonacci numbers and whose area is non-zero. In fact, while you are at it, consider several.

Also consider the possibility that the area, perimeter, and altitude to the base might be expressible in terms of Fibonacci numbers.

It doesn't take long to discover that all the triangles under consideration are isosceles. Further, it is obvious that the perimeter is already expressed in terms of Fibonacci numbers. But what about the area and the altitude to the base?

To aid and accomplish this end, it is suggested that Hero's formula for finding the area of a triangle be used. That is:

$$A = \sqrt{S(S - a)(S - b)(S - c)} ,$$

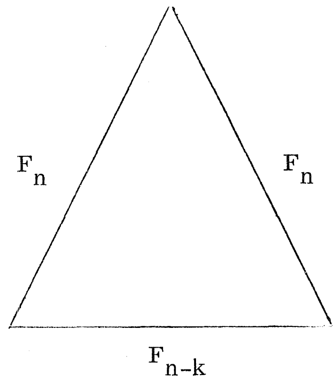
where

$$S = \frac{1}{2}(a + b + c)$$

if a, b, and c are the sides of the triangle.

Before continuing, it might be helpful to classify the triangles into some general categories and thus avoid random samples. The aim here, should it be attempted before reading on, would be to classify ALL the Fibonacci triangles.

The form selected here will be where the sides have length F_n, F_n, F_{n-k} with k an integer. Thus the groups of triangles might be represented as follows:



$k = -1$	$k = 0$	$k = 1$
2, 2, 3	1, 1, 1	2, 2, 1
3, 3, 5	2, 2, 2	3, 3, 1
5, 5, 8	3, 3, 3	5, 5, 3
etc.	etc.	etc.
$k = 2$	$k = 3$	$k = 4$
3, 3, 1	5, 5, 1	8, 8, 1
5, 5, 2	8, 8, 2	13, 13, 2
8, 8, 3	13, 13, 3	21, 21, 3
etc.	etc.	etc.

and so on for $k \geq -1$.

Now for an intuitive idea of what the area of a Fibonacci triangle might be for any k.

For k = -1,

SIDES		AREA (using Hero's formula)	
2, 2, 3	F_3, F_3, F_4	$\frac{3}{4}\sqrt{1 \cdot 7}$	$\frac{F_4}{L_3}\sqrt{F_1 L_4}$
3, 3, 5	F_4, F_4, F_5	$\frac{5}{4}\sqrt{1 \cdot 11}$	$\frac{F_5}{L_3}\sqrt{F_2 L_5}$
5, 5, 8	F_5, F_5, F_6	$\frac{8}{4}\sqrt{2 \cdot 18}$	$\frac{F_6}{L_3}\sqrt{F_3 L_6}$
8, 8, 13	F_6, F_6, F_7	$\frac{13}{4}\sqrt{3 \cdot 29}$	$\frac{F_7}{L_3}\sqrt{F_4 L_7}$
13, 13, 21	F_7, F_7, F_8	$\frac{21}{4}\sqrt{5 \cdot 47}$	$\frac{F_8}{L_3}\sqrt{F_5 L_8}$
Without using F_n, F_n, and F_{n-k} in the formula, what is the area in general?	$\vdots$ F_n, F_n, F_{n+1}		$\vdots$ $\frac{F_{n+1}}{L_3}\sqrt{F_{n-2}L_{n+1}}$

Once again, the generalization is found by looking at these two columns. What APPEARS to be the relationship between the specific example and its answer — then generalize. It will be seen later how this can be proved, or really how it can be verified as one expression for the area.

The reader may now wish to complete the following:

$k = 0$

	F_n, F_n, F_n		$\frac{F_n^2}{L_3}\sqrt{F_4}$

$k = 1$

	F_n, F_n, F_{n-1}		$\frac{F_{n-1}}{L_3}\sqrt{L_{n-1}F_{n+2}}$

$k = 2$

	$F_n,\ F_n,\ F_{n-2}$		$\frac{F_{n-2}}{L_3}\sqrt{F_{n+1}(F_{n-4} + L_{n+1})}$

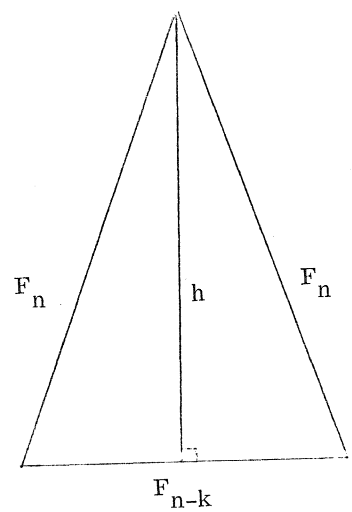
$k = 3$

	$F_n,\ F_n,\ F_{n-3}$		$\frac{F_{n-3}}{L_3}\sqrt{L_n(F_{n-4} + F_{n+1})}$

There is no need to stop at $k = 3$ and the interested reader may later with to continue. However, temporarily put aside the information found so far.

The Pythagorean Theorem and the fact that the base of an isosceles triangle is bisected by the altitude to that base leads to:

$$h^2 + \left(\frac{1}{2} F_{n-k}\right)^2 = F_n^2$$



or

$$h^2 = \frac{4F_n^2 - F_{n-k}}{4}$$

and

$$h = \frac{1}{2}\sqrt{4F_n^2 - F_{n-k}^2}$$

or

$$h = \frac{\sqrt{F_3^2 F_n^2 - F_{n-k}^2}}{F_3}$$

Then since $A = 1/2\ bh$,

$$A = \frac{F_{n-k}}{L_3}\sqrt{F_3^2 F_n^2 - F_{n-k}^2}$$

It is unlikely that this was what was arrived at using Hero's formula — and it might prove interesting to equate the two at this time. That is, for $k = -1$,

$$A = \frac{F_{n+1}}{L_3}\sqrt{F_{n-2} L_{n+1}}$$

using Hero's formula

$$A = \frac{F_{n+1}}{L_3}\sqrt{F_3^2 F_n^2 - F_{n+1}^2}$$

using $A = 1/2\ bh$. Therefore,

$$\frac{F_{n+1}}{L_3}\sqrt{F_{n-2}L_{n+1}} = \frac{F_{n+1}}{L_3}\sqrt{F_3^2 F_n^2 - F_{n+1}^2}$$

or

$$F_{n-2}L_{n+1} = F_3^2 F_n^2 - F_{n+1}^2$$

and

$$L_{n+1} = \frac{F_3^2 F_n^2 - F_{n+1}^2}{F_{n-2}} ,$$

which is the same as

$$\boxed{L_n = \frac{F_3^2 F_{n-1}^2 - F_n^2}{F_{n-3}}}$$

Thus, a new method for obtaining identities.

Although $k = 0$ is rather uninteresting, the reader may now wish to check and see what identities are produced by other k's. The results can be proved by conventional methods.

Anyone for Lucas triangles?