A SERIES FORM FOR THE FIBONACCI NUMBERS F_{12n}

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ABSTRACT

The Fibonacci Numbers, F_{12n}, have been found to be expressible as a series:

$$F_{12(2m-1)} = (2m - 1)F_{12}\sum_{j=1}^{m}\left[\frac{C(m - 1 + j, 2j - 1)}{(m - 1 + j)}(5 \times 12^4)^{j-1}\right]$$

and

$$F_{12(2m)} = F_{24}\sum_{j=1}^{m}\left[C(m - 1 + j, 2j - 1)(5 \times 12^4)^{j-1}\right],$$

where m is the running index and C(p,q) is the combination of p and q. The rationale by which these series were derived is given: namely, by writing F_k in the basic twelve system, recognizing groupings among the digits, and writing a summation series for the corresponding groups within sequential numbers.

1. INTRODUCTION

A list of 571 Fibonacci numbers, F_k, given by Basin and Hoggatt [1] shows that F_1 is 1 or 1^2 and F_{12} is 144 or 12^2. No other such coincidences were found, at least for the second power. Other relations will be shown below for F_{12n}, where n is an integer.

If d_i is the digit in the 10^{i-1} place, then there are cyclic relations among the d_i. That is, d_1 of F_{k+60} = d_1 of F_k, and d_2 of F_{k+300} = d_2 of F_k. The cycling period for d_3 is not to be found from the numbers in the above table. However, see Hoggatt [2].

To further the study of F_{12n} let us examine Table 1 in which F_k is written in the base twelve. The first 72 Fibonacci numbers are shown with X and e representing the tenth and eleventh digits, respectively. If D_i is the digit in the 12^{i-1} place, it is evident that D_1 of $F_{k+24} = D_1$ of F_k, and D_2 of $F_{k+24} = D_2$ of F_k. By examining an expansion of Table 1, one finds that D_3 of $F_{k+288} = D_3$ of F_k. These cyclic relations suggest that the digits change in shorter cycles in the base twelve than in the base ten. Therefore, a sequence of digits might be more readily recognized in the base twelve than in the base ten. This paper presents the rationale by which the particular series were found.

Table 1

THE FIRST 61 FIBONACCI NUMBERS IN THE BASE TWELVE

k	F_k	k	F_k	k	F_k
1	1	25	37,501	49	1,611,102,X01
2	1	26	5X,301	50	2,533,148,601
3	2	27	95,802	51	3,e44,24e,402
4	3	28	133,e03	52	6,477,397,X03
5	5	29	209,705	53	X,3ee,627,205
6	8	30	341,608	54	14,876,X03,008
7	11	31	54e,111	55	23,076,42X,211
8	19	32	890,719	56	37,931,231,219
9	2X	33	1,21e,82X	57	5X,9X7,65e,42X
10	47	34	1,Xe0,347	58	96,718,890,647
11	75	35	3,10e,e75	59	135,504,32e,X75
12	100	36	5,000,300	60	210,021,000,500
13	175	37	8,110,275	61	345,525,330,375
14	275	38	11,110,575	62	555,546,330,875
15	42X	39	19,220,82X	63	89X,X6e,661,02X
16	6X3	40	2X,331,1X3	64	1,234,3e5,991,8X3
17	e11	41	47,551,X11	65	1,e13,265,432,911
18	1,5e4	42	75,882,ee4	66	3,147,65e,204,5e4
19	2,505	43	101,214,X05	67	5,05X,904,637,305
20	2,Xe9	44	176,X97,9e9	68	8,1X6,363,83e,8e9
21	6,402	45	278,0e0,802	69	11,245,068,277,002
22	X,2ee	46	432,e88,5ee	70	19,42e,40e,Xe6,8ee
23	14,701	47	6Xe,079,201	71	2X,674,478,171,901
24	22,X00	48	e22,045,800	72	47,XX3,888,068,600

2. GROUPS OF FOUR DIGITS

Certain F_k's have been extracted from Table 1 and its logical extension to form Tables 2(a) and 3(a); F_{12n} are shown where n is an odd integer in Table 2(a) and an even integer in Table 3(a).

The smaller F_k's contain a surprising number of zeros so that they almost naturally fall into groups of four digits. For example, in Table 2(a),

$$F_{60} = 210021000500 \quad \text{or} \quad 21 - 0021 - 0005 - 00$$

in which the four groups contain the small integers 21, 21, 5, and 0. When written in the base ten,

$$F_{60} = 25 \times 12^{10} + 25 \times 12^{6} + 5 \times 12^{2} .$$

Table 2(a)

FIBONACCI NUMBERS IN BASE TWELVE

n	F_{12n}	j=9	8	7	6	5	4	3	2	1		Row R
1	F_{12}									1	00	1
3	F_{36}								5	0003	00	2
5	F_{60}							21	0021	0005	00	3
7	F_{84}						X5	0127	005X	0007	00	4
9	F_{108}					441	0799	0483	0106	0009	00	5
11	F_{132}				1985	3e83	3224	1145	01Xe	000e	00	6
13	F_{156}			9062	e616	e615	e350	2772	031e	0011	00	7
15	F_{180}	3	9274	3784	6922	3571	8676	5576	04X4	0013	00	8

Table 2(b)

n	F_{12n}	j=9	8	7	6	5	4	3	2	1		Row R
13	F_{156}			9061	1e615	1e615	e350	2772	031e	0011	00	7
15	F_{180}		39265	e3773	116916	8356e	28676	5576	04X4	0013	00	8

__ "Borrowed" integers

Table 3(a)
FIBONACCI NUMBERS IN BASE TWELVE

n	F_{12n}	j=7	6	5	4	3	2	1		Row R
2	F_{24}									1
4	F_{48}						e22	22X	00	2
6	F_{72}					47XX	3888	0686	00	3
8	F_{96}			1	e364	3e50	9398	08e4	00	4
10	F_{120}		9	8581	6420	19e7	6774	0e22	00	5
12	F_{144}	40	6463	0821	X679	8X86	873X	1150	00	6

Table 3(b)

n	F_{12n}	j=7	6	5	4	3	2	1		Row R
8	F_{96}				1e362	$\underline{2}$3e50	9398	08e4	00	4
10	F_{120}			$\underline{9}$856X	$\underline{1}$36414	$\underline{8}$19e6	$\underline{1}$6774	0e22	00	5
12	F_{144}		$\underline{4}$063X2	$\underline{8}$10784	$\underline{5}$9X660	$\underline{1}$98X84	$\underline{2}$873X	1150	00	6

$\underline{\quad}$ "Borrowed" integers

Form Table 4(a) from Table 2(a) by writing the integers within each group in the base ten. In writing this table, certain numbers were "borrowed" from one group to expand a following group. (The "borrowed" digits are indicated in Table 2(b) by underlining.)

Likewise, form Table 5(a) from Table 3(a) using "borrowed" integers as shown in Table 3(b) as necessary. Further, since F_{24n}/F_{24} is an integer, form Table 5(b) by dividing all groups by 322 which is the only group in F_{24} ($F_{24} = 322 \times 12^2$). Tables 4(a) and 5(b) are similar, especially the heads of the columns — 1, 5, 25, 125, etc., which are powers of 5. Of course, the integer five plays a large role in the expression for Fibonacci numbers

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

so one should not be surprised to find the digit 5 prominent in any expression for F_k.

By dividing each column by the number at its head, we obtain Table 4(b) from Table 4(a) and Table 5(c) from Table 5(b). Again, this process requires that digits be "borrowed" as noted above.

Table 4(a)

FIBONACCI NUMBERS IN BASE TEN WITH SPECIAL NOTATION

n	F_n	j= 8	7	6	5	4	3	2	1	Row R
		$x12^{30}$	$x12^{26}$	$x12^{22}$	$x12^{18}$	$x12^{14}$	$x12^{10}$	$x12^{6}$	$x12^{2}$	
1	F_{12}								1	1
3	F_{36}							5	3	2
5	F_{60}						25	25	5	3
7	F_{84}					125	175	70	7	4
9	F_{108}				625	1,125	675	150	9	5
11	F_{132}			3,125	6,875	5,500	1,925	275	11	6
13	F_{156}		15,625	40,625	40,625	19,500	4,550	455	13	7
15	F_{180}	78,125	234,375	281,250	171,875	66,250	9,450	700	15	8
		Table 4(b)								
		$x12^{30}$ $x5^{7}$	$x12^{26}$ $x5^{6}$	$x12^{22}$ $x5^{5}$	$x12^{18}$ $x5^{4}$	$x12^{14}$ $x5^{3}$	$x12^{10}$ $x5^{2}$	$x12^{6}$ $x5^{1}$	$x12^{2}$ $x5^{0}$	
1	F_{12}								1	1
3	F_{36}							1	3	2
5	F_{60}						1	5	5	3
7	F_{84}					1	7	14	7	4
9	F_{108}				1	9	27	30	9	5
11	F_{132}			1	11	44	77	55	11	6
13	F_{156}		1	13	65	156	182	91	13	7
15	F_{180}	1	15	90	275	450	378	140	15	8

3. SERIES FORMS

The rows in Tables 4 and 5 will be designated by R and the columns by j counting from right to left. For Table 5(c) the integers in each group are the binomial coefficients or the combinations $C(R-1+j,\ 2j-1)$. For Table 4(b), the integers in each group are not the binomial coefficient or the combinations, but can be so expressed when a multiplier and a divisor are included: the expression is

$$\frac{(2R-1)C(R-1+j, 2j-1)}{(R-1+j)} .$$

Table 5(a)

FIBONACCI NUMBERS IN BASE TEN WITH SPECIAL NOTATION

n	F_n	j= 6	5	4	3	2	1	Row R
		$\times 12^{22}$	$\times 12^{18}$	$\times 12^{14}$	$\times 12^{10}$	$\times 12^{6}$	$\times 12^{2}$	
2	F_{24}						322	1
4	F_{48}					1,610	644	2
6	F_{72}				8,050	6,440	966	3
8	F_{96}			40,250	48,300	16,100	1,288	4
10	F_{120}		201,250	322,000	169,050	32,200	1,610	5
12	F_{144}	1,006,250	2,012,500	1,449,000	450,800	56,350	1,932	6

Table 5(b)

n	F_n	j= 6	5	4	3	2	1	Row R
		$\times 12^{22}$ $\times 322$	$\times 12^{18}$ $\times 322$	$\times 12^{14}$ $\times 322$	$\times 12^{10}$ $\times 322$	$\times 12^{6}$ $\times 322$	$\times 12^{2}$ $\times 322$	
2	F_{24}						1	1
4	F_{48}					5	2	2
6	F_{72}				25	20	3	3
8	F_{96}			125	150	50	4	4
10	F_{120}		625	1,000	525	100	5	5
12	F_{144}	3,125	6,250	4,500	1,400	175	6	6

Table 5(c)

n	F_n	j= 6	5	4	3	2	1	Row R
		$\times 12^{22}$ $\times 322$ $\times 5^{5}$	$\times 12^{18}$ $\times 322$ $\times 5^{4}$	$\times 12^{14}$ $\times 322$ $\times 5^{3}$	$\times 12^{10}$ $\times 322$ $\times 5^{2}$	$\times 12^{6}$ $\times 322$ $\times 5^{1}$	$\times 12^{2}$ $\times 322$ $\times 5^{0}$	
2	F_{24}						1	1
4	F_{48}					1	2	2
6	F_{72}				1	4	3	3
8	F_{96}			1	6	10	4	4
10	F_{120}		1	8	21	20	5	5
12	F_{144}	1	10	36	56	35	6	6

Of course, F_k is the sum of the numbers in a row multiplied by the proper factors for each column. It is convenient to write 12^2 as F_{12} and $322\ F_{12}$ as F_{24}. One notes that in summing, j runs from 1 to R in every case. In Table 2(a), n = 2R - 1, and in Table 3(a), n = 2R. Therefore, for the odd multiples of 12, one has

$$F_{12(2R-1)} = (2R - 1)F_{12}\sum_{j=1}^{R}\left[\frac{C(R - 1 + j,\ 2j - 1)}{(R - 1 + j)}\ 5^{j-1} \times 12^{4(j-1)}\right]$$

and for the even multiples of 12, one has

$$F_{12(2R)} = F_{24}\sum_{j=1}^{R}\left[C(R - 1 + j,\ 2j - 1) \times 5^{j-1} \times 12^{4(j-1)}\right].$$

5. SUMMARY

(A) d_i for F_k occur in cycles. The cycles of d_i are shorter when F_k is written in the base twelve than when F_k is written in the base ten.

(B) F_{12n} written in the base twelve may be split into groups of four digits. Some borrowing among groups may be needed for the larger numbers to retain integers in the groups.

(C) When the groups are rewritten in the base ten, certain features stand out: (1) For n odd, F_{12n}/F_{12} is the integer shown in Table 4(a). (2) For n even, F_{12n}/F_{24} is the integer shown in Table 5(b). (3) Each column of integers is divisible by a power of five given by 5^{j-1} where j is the number of the column counting from right to left. (4) The quotients left after dividing by 5^{j-1} are expressible as combinations and factors using the row and column designators. (5) F_k is formed by summing the numbers in its row multiplied by the proper factors for each column.

(D) F_{12n} may be expressed by the summation series that are given in the Abstract.

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[Continued from page 404.]

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[Continued on page 436.]