ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-186 Proposed by James Desmond, Florida State University, Tallahasse, Florida.

The generalized Fibonacci sequence is defined by the recurrence relation

$$U_{n-1} + U_n = U_{n+1},$$

where n is an integer and U_0 and U_1 are arbitrary fixed integers.

For a prime p and integers n, r, s and t show that

$$U_{np+r} \equiv U_{sp+t} \pmod p$$

if $p \equiv \pm 1 \pmod 5$ and $n + r = s + t$, and that

$$U_{np+r} \equiv (-1)^{r+t} U_{sp+t} \pmod p$$

if $p \equiv \pm 2 \pmod 5$ and $n - r = s - t$.

H-187 Proposed by Ira Gessel, Harvard University, Cambridge, Massachusetts.

Problem: Show that a positive integer n is a Fibonacci number if and only if either $5n^2 + 4$ or $5n^2 - 4$ is a square.

H-188 Proposed by Raymond E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Prove that there are no even perfect Fibonacci numbers.

SOLUTIONS

A NORMAL DETERMINANT

H-168 Proposed by David A. Klarner, University of Alberta, Edmonton, Alberta, Canada.

If

$$a_{ij} = \binom{i + j - 2}{i - 1}$$

for $i, j = 1, 2, \cdots, n$, show that $\det a_{ij} = 1$.

Solution by F. D. Parker, St. Lawrence University, Canton, New York.

It will be convenient to denote the given matrix by Mn, and its determinant by $d(Mn)$, and then to prove the result by mathematical induction.

Since

$$a_{ij} = \binom{i + j - 2}{i - 1} ,$$

we have the two identities

$$a_{ij} - a_{i-1,j} = a_{i,j-1} ,$$

and

$$a_{ij} - a_{i,j-1} = a_{i-1,j} .$$

If we subtract from each column (except the first) of Mn the preceding column, the second identity shows that

$$d(Mn) = d(C_{i1}, C_{i-1,2}, C_{i-1,3}, \cdots, C_{i-1,n}) ,$$

where c_{ij} represents a column whose elements are given by a_{ij}. We notice that the first row of this new matrix is $(1, 0, 0, \cdots)$. Now if we subtract from each row (except the first) of the new matrix the preceding row, the first identity produces the matrix

$$Mn'' = \begin{pmatrix} 1 & \overline{0} \\ I & M_{n-1} \end{pmatrix}$$

where $\overline{0}$ is a row vector of zeros, I is a column vector of ones. The determinant has not been changed by these operations so that we have

$$d(Mn) = d(Mn'') = d(Mn - 1) .$$

Thus $d(Mn)$ is a constant and, since $d(m1) = 1$, then $d(Mn) = 1$.

Also solved by C. B. A. Peck and M. Yoder.

PRIME TARGET

H-169 Proposed by Francis DeKoven, Highland Park, Illinois. (Correction).

Show $n^2 + 1$ is a prime if and only if $n \neq ab + cd$ with $ad - bc = \pm 1$ for integers $a, b, c, d > 0$.

Solution by Robert Guili, San Jose State College, San Jose, California. (Partial)

Note: Z denotes the set of positive integers.

Solution by contradiction: If

$$n = ab + cd; \qquad ad - bc = \pm 1 ,$$

then

$$n^2 = a^2b^2 + 2abcd + c^2d^2 ; \qquad 1 = a^2d^2 - 2abcd + b^2c^2 .$$

So

$$\begin{aligned} n^2 + 1 &= a^2b^2 + a^2d^2 = c^2d^2 + b^2c^2 \\ &= a^2(b^2 + d^2) + c^2(d^2 + b^2) \\ &= (a^2 + c^2)(b^2 + d^2) \end{aligned}$$

which is not true.

EDITORIAL COMMENT

The second part of this proof intended here was not complete. The late proposer made the same logical oversight. However, the second proof he submitted was more complete and can appear at a later date.

Editor V. E. H.

Also solved by the Proposer.

NON-EXISTENT

H-171 Proposed by Douglas Lind, Stanford University, Stanford, California.

Does there exist a continuous real-valued function f defined on a compact interval I of the real line such that

$$\int_I f(x)^n \, dx = F_n \; .$$

What if we require f only be measurable?

Solution by the Proposer.

We claim that such a measurable function f does not exist. By the Binet formula,

$$F_n = (a^n - b^n)/\sqrt{5} ,$$

where

$$a = (1 + \sqrt{5})/2, \qquad b = (1 - \sqrt{5})/2 .$$

For any measurable real-valued function g defined on I and any $p \geq 1$ we define

$$\|g\|_{p,I} \equiv \|g\|_p = \left(\int_I |g(x)|^p dx\right)^{1/p}$$

which is taken to be $+\infty$ if $|g|^p$ is not Lebesgue integrable. Also, let

$$\|g\|_{\infty,I} \equiv \|g\|_\infty = \text{ess sup}\{|g(x)|;\ x \in I\} = \inf\{t: \mu(g^{-1}(t,\infty)) = 0\} ,$$

where μ denotes Lebesgue measure on the real line. It is well known that since $\mu(I) < \infty$,

$$\lim_{p\to\infty} \|g\|_p = \|g\|_\infty ,$$

where $\|g\|_\infty$ is possibly ∞.

Now suppose that f is a real-valued function on I such that

$$F_n = \int_I f^n(x)\,dx$$

for $n = 1, 2, \cdots$. Then

$$\|f\|_\infty = \lim_{n\to\infty} \|f\|_n = \lim_{n\to\infty} F_n^{1/n} = a .$$

Let

$$A = \{x \in I: f(x) = a\} ,$$

$$B = \{x \in I: f(x) = -a\} .$$

Then for $n = 2k$ we have

$$\frac{a^{2k} - b^{2k}}{\sqrt{5}} = \int_I f^{2k}(x)dx = \{\mu(A) + \mu(B)\}a^{2k} + \int_{I-(A\cup B)} f^{2k}(x)dx ,$$

so that

$$(*) \qquad \frac{1}{\sqrt{5}} - \mu(A) - \mu(B) = \frac{1}{\sqrt{5}}\left(\frac{b}{a}\right)^{2k} + \int_{I-(A\cup B)} \left[\frac{f(x)}{a}\right]^{2k} dx .$$

Since $|f(x)/a| < 1$ for almost all $x \in I - (A\cup B)$,

$$\{f(x)/a\}^{2k} \to 0$$

a.e. on $I - (A\cup B)$ as $k \to \infty$, so by Lebesgue's Dominated Convergence Theorem, the right-hand integral approaches 0 as $k \to \infty$. Since

$$|b/a| < 1, \qquad (b/a)^{2k} \to 0$$

as $k \to \infty$, so letting $k \to \infty$ in (*) shows

$$\mu(A) + \mu(B) = 1/\sqrt{5} .$$

Now if we put $n = 2k + 1$, we have

$$\frac{a^{2k+1} - b^{2k+1}}{\sqrt{5}} = \{\mu(A) - \mu(B)\}a^{2k+1} + \int_{I-(A\cup B)} f^{2k+1}(x)dx ,$$

and the same reasoning as before shows

$$\mu(A) - \mu(B) = 1/\sqrt{5} ,$$

Hence $\mu(B) = 0$ and $\mu(A) = 1/\sqrt{5}$. Letting $K = I - A$, we thus have

$$\frac{-b^n}{\sqrt{5}} = \int_K f^n(x)dx \ .$$

Now

$$|b| = \lim_{n\to\infty} \|f\|_{n,K} = \|f\|_{\infty,K} \ ,$$

so

$$\|f(x)\| \leq |b|$$

for almost all $x \in K$. Let

$$C = \{x \in K: f(x) = b\} \ ,$$

$$D = \{x \in K: f(x) = -b\} \ .$$

Then

$$\frac{-b^{2k}}{\sqrt{5}} = \{\mu(C) + \mu(D)\}b^{2k} + \int_{K-(C\cup D)} f^{2k}(x)dx \ ,$$

so that

$$\frac{1}{\sqrt{5}} + \mu(C) + \mu(D) = -\int_{K-(C\cup D)} \left[\frac{f(x)}{b}\right]^{2k} dx \ .$$

Reasoning as before, we see by dominated convergence that the right-hand integral approaches 0 as $k \to \infty$. But this contradicts the fact that the left side is strictly positive. This contradiction shows that such an f does not exist.

We remark that the situation is different for Lucas numbers. For let $1 = [0,2]$, $f(x) = a$ if $0 \leq x < 1$, $f(x) = b$ if $1 \leq x \leq 2$. Then

$$\int_I f^n(x)dx = a^n + b^n = L_n .$$

However, one can show using the methods above that f cannot be replaced by a continuous function.

Editorial Note: Robert Giuli noted that

$$\int_b^a \frac{nx^{n-1}}{\sqrt{5}} dx = F_n ,$$

although this does not satisfy the proposal. It might be interesting to reconsider the proposal with restrictions on f, such as boundedness, etc.

HISTORY REPEATS

H-172 Proposed by David Englund, Rockford College, Rockford, Illinois.

Prove or disprove the "identity,"

$$F_{kn} = F_n \sum_{t=1}^{\left[\frac{k+1}{2}\right]} (-1)^{(n+1)(t+1)} \binom{k-t}{t-1} L_n^{k-2t+1} .$$

where F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively, and $[x]$ denotes the greatest integer function.

Solution by Douglas Lind, Stanford University

This is Problem H-135 (this Quarterly, Vol. 6, 1968, pp. 143-144: solution, Vol. 7, 1969, pp. 518-519), and appears as Eq. (3.15) in "Compositions and Fibonacci Numbers" by V. E. Hoggatt, Jr., and D. A. Lind (this Quarterly, Vol. 7, 1969, pp. 253-266).

Also solved by Wray Brady and L. Carlitz.

FIBONACCI VERSUS DIOPHANTUS

H-173 Proposed by George Ledin, Jr., Institute of Chemical Biology, University of San Francisco, San Francisco, California

Solve the Diophantine equation,

$$x^2 + y^2 + 1 = 3xy \; .$$

Solution by L. Carlitz, Duke University, Durham, North Carolina.

The equation

$$(*) \qquad x^2 + y^2 + 1 = 3xy$$

can be written in the form

$$(ax - 3y)^2 - 5y^2 = -4 \; ,$$

where $a = 2$. We recall that the general (positive) solution of

$$x^2 - 5y^2 = -4$$

is given by

$$\left(\frac{1+\sqrt{5}}{2}\right)^{2n+1} = \frac{u_n + v_n\sqrt{5}}{2} \qquad (n = 0, 1, 2, \cdots) \; ,$$

so that

$$\begin{cases} u_n = \dfrac{1}{2^{2n}} \displaystyle\sum_{r=0}^{n} \binom{2n+1}{2r} 5^r \\ \\ v_n = \dfrac{1}{2^{2n}} \displaystyle\sum_{r=0}^{n} \binom{2n+1}{2r+1} 5^r \; . \end{cases}$$

On the other hand, the Fibonacci number F_{n+1} satisfies

$$F_{n+1} = \frac{1}{2^n} \sum_{2r \le n} \binom{n+1}{2r+1} 5^r ,$$

so that $v_n = F_{2n+1}$. Moreover,

$$u_n + v_n = 2F_{2n+2} ,$$

which gives

$$u_n = 2F_{2n+2} - F_{2n+1} .$$

Since

$$y = v_n, \quad 2x - 3y = u_n ,$$

it follows that

$$2x = u_n + 3v_n = 2F_{2n+2} + 2F_{2n+1} = 2F_{2n+3} ,$$

so that $x = F_{2n+3}$. Hence we have the general solution of (*) with $x > y$:

$$x = F_{2n+3}, \quad y = F_{2n+1} \qquad (n = 0, 1, 2, \cdots) .$$

The solution $x = y = 1$ is evidently obtained by taking $n = -1$.

Also solved by W. Barley, M. Herdy, C. B. A. Peck, C. Bridger, J. A. H. Hunter, and the Proposer.

SUM PROJECT

H-175 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$(1 + z + \tfrac{1}{3}z^2)^{-n-1} = \sum_{k=0}^{\infty} a(n,k)z^k \quad .$$

Show that

$$\text{(1)} \qquad a(n,n) = \frac{2\cdot 5\cdot 8 \cdots (2n - 1)}{n!}$$

$$\text{(II)} \qquad \sum_{s=0}^{n} \binom{n - s}{s}\binom{2n - s}{n}\left(-\frac{1}{3}\right)^s = \frac{2\cdot 5\cdot 8 \cdots (3n - 1)}{n!}$$

$$\text{(III)} \qquad \sum_{r=0}^{\infty} \binom{n + r}{r}\binom{2n - r}{n}(-\omega)^r = (\omega^2\sqrt{-3})^n \frac{2\cdot 5\cdot 8 \cdots (3n - 1)}{n!} \quad ,$$

where

$$\omega = \frac{1}{2}(-1 - \sqrt{-3}) \quad .$$

Solution by the Proposer.

(I) If $z = wf(z)$, $f(0) \neq 0$, where $f(z)$ is analytic about the origin, then (Polya-Szegö, Aufgaben und Lehrsatze aus der Analysis, Vol. 1, p. 125)

$$z = \sum_{n=1}^{\infty} \frac{w^n}{n!}\left[\frac{d^{n-1}}{dx^{n-1}}(f(x))^n\right]_{x=0}$$

$$= \sum_{n=0}^{\infty} \frac{w^{n+1}}{(n+1)!}\left[\frac{d^n}{dx^n}(f(x))^{n+1}\right]_{x=0} \quad .$$

Take

$$f(z) = (1 - z + \tfrac{1}{3}z^2)^{-1} ,$$

so that

$$(*) \qquad \left[\frac{d^n}{dx^n}(f(x))^{n+1}\right]_{x=0} = n!\, a(n,n) .$$

On the other hand, $z = wf(z)$ becomes

$$z(1 - z + \tfrac{1}{3}z^2) = w ,$$

which reduces to

$$(1 - z)^3 = 1 - 3w .$$

It follows that

$$\begin{aligned} z &= 1 - (1 - 3w)^{\frac{1}{3}} \\ &= \sum_{n=1}^{\infty} (-1)^n \binom{-\frac{1}{3}}{n} 3^n w^n \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{2}{3}}{n} \frac{3^n w^n}{n + 1} \\ &= \sum_{n=0}^{\infty} \frac{2\cdot 5\cdot 8 \cdots (3n - 1)}{(n + 1)!} w^n . \end{aligned}$$

Comparison with (*) gives

$$a(n,n) = \frac{2\cdot 5\cdot 8 \cdots (3n - 1)}{n!} .$$

(II). Since

$$(1 - z + \tfrac{1}{3}z^2)^{-n-1} = \sum_{r=0}^{\infty} \binom{n+r}{r} z^r (1 - \tfrac{1}{3}z)^r$$

$$= \sum_{r=0}^{\infty} \binom{n+r}{r} z^r \sum_{s=0}^{r} \binom{r}{s} \left(-\tfrac{1}{3}\right)^s z^s$$

$$= \sum_{k=0}^{\infty} z^k \sum_{r+s=k} \binom{n+r}{r}\binom{r}{s}\left(-\tfrac{1}{3}\right)^s ,$$

it follows that

$$a(n,n) = \sum_{s=0}^{n} \binom{n-s}{s}\binom{2n-s}{n}\left(-\tfrac{1}{3}\right)^s .$$

(III). Put

$$1 - z + \tfrac{1}{3}z^2 = (1 - \alpha z)(1 - \beta z) .$$

It is easily verified that

$$\alpha = -\frac{\omega^2}{\sqrt{-3}}, \qquad \beta = \frac{\omega}{\sqrt{-3}} .$$

Then

$$(1 - z + \tfrac{1}{3}z^2)^{-n-1} = (1 - \alpha z)^{-n-1}(1 - \beta z)^{-n-1}$$

$$= \sum_{r=0}^{\infty} \binom{n+r}{r} \alpha^r z^r \sum_{s=0}^{\infty} \binom{n+s}{s} \beta^s z^s ,$$

so that

$$a(n,n) = \sum_{r+s=n} \binom{n+r}{r}\binom{n+s}{s}\alpha^r \beta^s$$

$$= \sum_{r=0}^{\infty} \binom{n+r}{r}\binom{2n-r}{n}\left(-\frac{\omega^2}{\sqrt{-3}}\right)^r \left(\frac{\omega}{\sqrt{-3}}\right)^{n-r}$$

$$= \frac{\omega^n}{(\sqrt{-3})^n} \sum_{r=0}^{n} \binom{n+r}{r}\binom{2n-r}{n}(-\omega)^r .$$

[Continued from page 496.]

GENERALIZED BASES FOR REAL NUMBERS

3. S. Kakeya, "On the Partial Sums of an Infinite Series," Sci. Reports Tohoku Imp. U. (1), 3 (1914), pp. 159-163.
4. J. L. Brown, Jr., "On the Equivalence of Completeness and Semi-Completeness for Integer Sequences," Mathematics Magazine, Vol. 36, No. 4, Sept.-Oct., 1963, pp. 224-226.
5. I. Niven, "Irrational Numbers," Carus Mathematical Monograph No. 11, John Wiley and Sons, Inc., 1956.
6. I. Niven and H. S. Zuckerman, An Introduction to the Theory of Numbers, John Wiley and Sons, Inc., 1960.

CHALLENGE

"In what way does the cubic congruence

$x^3 - 15x + 25 \equiv 0 \pmod{p}$, p a prime

relate to the Fibonacci numbers?

Generalize to other recurring series."

John Brillhart and Emma Lehmer