ON THE COEFFICIENTS OF A GENERATING SERIES

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1. INTRODUCTION

Our object of study is the generating series

$$\prod_{n=1}\left(1 + \rho x^{u_n}\right) = \sum^{\infty} \epsilon(n) x^n \quad , \tag{1}$$

where the coefficients $\epsilon(n)$ are polynomials in ρ, and where $\{u_n\}$ is the sequence defined by

$$u_1 = 1, \quad u_2 = 2, \quad u_n = u_{n-1} + u_{n-2} \quad \text{for} \quad n > 2 \ . \tag{2}$$

Theorem 1. The values assumed by the coefficients $\epsilon(n)$ as $n = 0, 1, 2, \cdots$ range over a finite set if and only if ρ is one of the numbers 0, -1, ω, or ω^2, where ω and ω^2 are the complex cube roots of unity.

The theorem has applications to partition theory. It implies the existence of certain symmetries, which we illustrate in Section 5, among the partitions of integers into terms of the sequence $\{u_n\}$. Sections 3 and 4 are devoted to the proof of Theorem 1. In Section 2, some preliminary recursion formulas are obtained, which find application in Sections 3 and 4.

For an added comment, see note at conclusion of this article.

2. RECURSION FORMULAS FOR $\epsilon(n)$

For each natural number n, let $\nu(n)$ denote the largest index k for which $u_k \le n$. Thus $\nu(n)$ is defined by the condition that

$$u_{\nu(n)} \le n < u_{\nu(n)+1} \ . \tag{3}$$

Writing $\epsilon(m) = 0$ for negative m, we prove that

Lemma 1. For $n > 1$, $\epsilon(n)$ satisfies the recursion

$$\epsilon(n) = \rho\epsilon(n - u_\nu) + \rho\epsilon(n - u_{\nu-1}) - \rho^2\epsilon(n - 2u_{\nu-1}) , \tag{4}$$

where we have written ν for $\nu(n)$.

For a fixed natural number n, write $f(x) = g(x)$ if $f(x)$ and $g(x)$ are formal power series whose difference contains only terms of degree greater than n. Then (1) and (3) imply that

$$\sum_{m=0}^{n} \epsilon(m)x^m \equiv \prod_{m=1}^{\nu(n)} \left(1 + \rho x^{u_m}\right) .$$

From (2) and (3) it follows that

$$\left(1 + \rho x^{u_\nu}\right)^{-1}\left(1 + \rho x^{u_{\nu-1}}\right)^{-1} \equiv 1 - \rho x^{u_\nu} - \rho x^{u_{\nu-1}} + \rho^2 x^{2u_{\nu-1}} .$$

so that

$$\left(1 - \rho x^{u_\nu} - \rho x^{u_{\nu-1}} + \rho^2 x^{2u_{\nu-1}}\right)\sum_{m=0}^{n} \epsilon(m)x^m = \prod_{m=1}^{\nu-2} \left(1 + \rho x^{u_m}\right).$$

Equating coefficients of x^n, we find that

$$\epsilon(n) - \rho\epsilon(n - u_\nu) - \rho\epsilon(n - u_{\nu-1}) + \rho^2\epsilon(n - 2u_{\nu-1}) \tag{5}$$

is the coefficient of x^n in

$$\prod_{m=1}^{\nu-2} \left(1 + \rho x^{u_m}\right) .$$

Now from the identity

$$\sum_{m=1}^{\nu-2} u_m = u_\nu - 2 ,$$

(an immediate consequence of (2)), it is clear that

$$\deg \prod_{m=1}^{\nu-2}\left(1 + \rho x^{u_m}\right) = u_\nu - 2 \le n - 2 ,$$

so that (5) vanishes, proving the lemma.

In sequel, a shall denote a natural number and we shall write σ for $\nu(a)$. From the inequalities

$$u_{\sigma+1} \le 2u_\sigma \le a + u_\sigma < u_{\sigma+1} + u_\sigma = u_{\sigma+2} ,$$
$$u_{\sigma+2} = u_\sigma + u_{\sigma+1} \le a + u_{\sigma+1} < 2u_{\sigma+1} \le u_{\sigma+3} ,$$
$$u_{\sigma+n} \le a + u_{\sigma+n} < u_{\sigma+1} + u_{\sigma+n} \le u_{\sigma+n+1} \quad \text{for} \quad n \ge 2 ,$$

we obtain

$$\nu(a + u_{\sigma+n}) = \begin{cases} \sigma + n + 1 & \text{if } 0 \le n < 2 \\ \sigma + n & \text{if } 2 \le n \end{cases} . \tag{6}$$

Applying the fundamental recursion (4),

$$\epsilon(a + u_\sigma) = \rho\epsilon(a - u_{\sigma-1}) + \rho\epsilon(a) - \rho^2\epsilon(a - u_\sigma) , \tag{7}$$

$$\epsilon(a + u_{\sigma+1}) = \rho\epsilon(a - u_\sigma) + \rho\epsilon(a) , \tag{8}$$
$$\epsilon(a + u_{\sigma+2}) = \rho\epsilon(a) + \rho\epsilon(a + u_\sigma) - \rho^2\epsilon(a - u_{\sigma-1})$$

from which it follows that

$$\epsilon(a + u_{\sigma+2}) = \rho(1 + \rho)\epsilon(a) - \rho^3\epsilon(a - u_\sigma) . \tag{9}$$

<u>Lemma 2.</u> For $h \ge 1$ and $\rho \ne 1$ we have

$$\epsilon(a + u_{\sigma+2h}) = \frac{\rho(1 - \rho^{h+1})}{1 - \rho}\,\epsilon(a) - \rho^{h+2}\epsilon(a - u_\sigma) . \tag{10}$$

For $k > 1$, Eq. (6) and Lemma 1 imply that

$$\epsilon(a + u_{\sigma+2k}) = \rho\epsilon(a) + \rho\epsilon(a + u_{\sigma+2k-2}) \tag{11}$$

since the term $\rho^2 \epsilon(a - u_{\sigma+2k-3})$ vanishes. Multiplying both sides of (11) by ρ^{-k} and summing,

$$\sum_{k=2}^{h} \rho^{-k}\epsilon(a + u_{\sigma+2k}) = \frac{\rho^{-1} - \rho^{-h}}{1 - \rho^{-1}}\epsilon(a) + \sum_{k=1}^{h-1} \rho^{-k}\epsilon(a + u_{\sigma+2k}) ,$$

so that, for $h \geq 2$,

$$\epsilon(a + u_{\sigma+2h}) = \frac{\rho(1 - \rho^{h-1})}{1 - \rho}\epsilon(a) + \rho^{h-1}\epsilon(a + u_{\sigma+2}) .$$

An appeal to (9) proves the lemma.

<u>Lemma 3.</u> For $h \geq 1$ and $\rho \neq 1$ we have

$$\epsilon(a + u_{\sigma+2h+1}) = \frac{\rho(1 - \rho^{h+1})}{1 - \rho}\epsilon(a) . \tag{12}$$

For $k > 1$, Eq. (16) and Lemma 1 imply that

$$\epsilon(a + u_{\sigma+2k+1}) = \rho\epsilon(a) + \rho\epsilon(a + u_{\sigma+2k+1}) .$$

Treating this in the same manner as (11), we get

$$\epsilon(a + u_{\sigma+2h+1}) = \frac{\rho(1 - \rho^{h-1})}{1 - \rho}\epsilon(a) + \rho^{h-1}\epsilon(a + u_{\sigma+3}) \tag{13}$$

for $h \geq 2$. But (6), (8) and Lemma 1 imply that

$$\epsilon(a + u_{\sigma+3}) = \rho\epsilon(a) + \rho\epsilon(a + u_{\sigma+1}) - \rho^2\epsilon(a - u_{\sigma}) = \rho(1 + \rho)\epsilon(a) .$$

Inserting this identity in (13), we arrive at (12), which is seen to hold for $h = 1$ as well.

3. NECESSITY THAT $\rho = 0, -1, \omega$, or ω^2

We can now prove that if the coefficients $\epsilon(1), \epsilon(2), \epsilon(3), \cdots$ range over a finite set of values, then ρ must be one of the numbers $0, -1, \omega$, or ω^2.

From (1) and (2), it is clear that $\epsilon(1) = \rho$ and $\nu(1) = 1$. Taking $a = 1$ and $\sigma = \nu(a) = 1$ in (12),

$$\epsilon(1 + u_{2h+2}) = \frac{\rho(1 - \rho^{h+1})}{1 - \rho}$$

for $h \geq 1$. If these values all lie in a finite set, then ρ must be either zero or a root of unity.

Taking $h = 1$ in (12), we get for $a \geq 0$,

$$\epsilon(a + u_{\sigma+3}) = \rho(1 + \rho)\epsilon(a) \quad . \tag{14}$$

Letting $a', a'', a''', \cdots$, and $\sigma', \sigma'', \sigma''', \cdots$ be defined by

$$\begin{aligned} a' &= a + u_{\sigma+3}, \quad \sigma' = \nu(a') \quad , \\ a'' &= a' + u_{\sigma'+3}, \quad \sigma'' = \nu(a'') \quad , \\ a''' &= a'' + u_{\sigma''+3}, \quad \sigma''' = \nu(a''') \quad , \end{aligned}$$

etc., we obtain by iterating (14),

$$\epsilon(a^{(t)}) = \rho^t(1 + \rho)^t\epsilon(a) \quad ;$$

since these values all lie in a finite set, $\rho(1 + \rho)$ must either be zero or a root of unity. Thus, either $\rho = 0$, $\rho = -1$, or both ρ and $1 + \rho$ are roots of unity, in which case it is a simple deduction that $\rho = \omega$ or $\rho = \omega^2$.

4. SUFFICIENCY OF $\rho = 0, -1, \omega$, or ω^2; THE METHOD OF DESCENT

If $\rho = 0$, it follows directly from (1) that $\epsilon(0) = 1$ and $\epsilon(n) = 0$ for $n \neq 0$. For the case $\rho = -1, \omega$, or ω^2, we shall employ a method of descent.

The next lemma is needed only for $\rho = \omega$ or ω^2. It is valid, however, for all ρ.

Lemma 4. For each natural number n, $\epsilon(n) - \rho\,\epsilon(n - u_\nu)$ either vanishes or is of the form $\rho^h \epsilon(m)$ for some $h \geq 0$ and some $m < n$.

We define a finite descending chain of natural numbers $n^{(0)} > n^{(1)} > n^{(2)} > \cdots$ as follows:

$$n^{(0)} = n, \quad \nu^{(0)} = \nu = \nu(n) .$$

If

$$n^{(k)} \leq 2u_{\nu^{(k)}-1} ,$$

the chain terminates at $n^{(k)}$; if, on the other hand,

$$n^{(k)} > 2u_{\nu^{(k)}-1} ,$$

define $n^{(k+1)}$ and $\nu^{(k+1)}$ by

$$n^{(k+1)} = n^{(k)} - u_{\nu^{(k)}-1} , \qquad \nu^{(k+1)} = \nu^{(k)} - 1 .$$

First, we show by induction on k that $\nu^{(k)} = \nu(n^{(k)})$, for if the chain extends to $n^{(k+1)}$, then

$$u_{\nu^{(k+1)}} = u_{\nu^{(k)}-1} < n^{(k)} - u_{\nu^{(k)}-1} = n^{(k+1)}$$

and

$$n^{(k+1)} = n^{(k)} - u_{\nu^{(k)}-1} < u_{\nu^{(k)}+1} - u_{\nu^{(k)}-1} = u_{\nu^{(k)}} = u_{\nu^{(k+1)}+1} .$$

Next, applying (4) to $n^{(k)}$, we arrive at

$$\epsilon(n^{(k)}) - \rho\epsilon(n^{(k)} - u_{\nu^{(k)}}) = \rho\left\{\epsilon(n^{(k+1)}) - \rho\epsilon(n^{(k+1)} - u_{\nu^{(k+1)}})\right\} ;$$

it follows that

$$\epsilon(n) - \rho\epsilon(n - u_{\nu}) = \rho^{k}\left\{\epsilon(n^{(k)}) - \rho\epsilon(n^{(k)} - u_{\nu^{(k)}})\right\} .$$

If $n^{(k)}$ is the last term in the chain, then (4) applied to $n^{(k)}$ yields

$$\epsilon(n^{(k)}) - \rho\epsilon(n^{(k)} - u_{\nu^{(k)}}) = \begin{cases} \rho\epsilon(m^{(k)} - u_{\nu^{(k)}-1}) & \text{if } n^{(k)} < 2u_{\nu^{(k)}-1} \\ \rho\left\{\epsilon(u_{\nu^{(k)}-1}) - \rho\right\} & \text{if } n^{(k)} = 2u_{\nu^{(k)}-1} \end{cases} .$$

Hence, in the first case,

$$\epsilon(n) - \rho\epsilon(n - u_{\nu}) = \rho^{k+1}\epsilon(n^{(k)} - u_{\nu^{(k)}-1}) .$$

Finally, (4) applied to u_t yields

$$\epsilon(u_t) = \rho + \rho\epsilon(u_{t-2}) ,$$

so that

$$\text{(15)} \qquad \epsilon(u_t) - \rho = \begin{cases} 0 \text{ if } t = 1 \text{ or } t = 2 \\ \rho\epsilon(u_{t-2}) \quad \text{otherwise} \end{cases} .$$

Therefore, the second case results in

$$\epsilon(n) - \rho\epsilon(n - u_{\ }) = \begin{cases} 0 \quad \text{if } \nu^{(k)} \le 3 \\ \rho^{k+2}\epsilon(u_{\nu^{(k)}-2}) \text{ otherwise} \end{cases} ,$$

and the lemma is proved.

Lemma 5. If $k \geq 2$ and $\rho = -1, \omega$, or ω^2, then $\epsilon(a + u_{\sigma+k})$ either vanishes or is of the form $\pm\rho^t\epsilon(m)$ for some $t \geq 0$ and some $m < a + u_{\sigma+k}$.

If k is odd, the result is a direct consequence of Lemma 3.

If k is even and $\rho = -1$, then Lemma 2 implies that $\epsilon(a + u_{\sigma+k})$ equals either $\epsilon(a - u_\sigma)$ or $-\epsilon(a) - \epsilon(a - u_\sigma)$ which, according to (8), in turn equals $\epsilon(a + u_{\sigma+1})$.

If is is even and $\rho = \omega$ or $\rho = \omega^2$, then Lemma 2 implies that

$$\epsilon(a + u_{\sigma+k}) = \begin{cases} \rho\left\{\epsilon(a) - \rho\epsilon(a - u_\sigma)\right\} & \text{if } k \equiv 0 \pmod 3 \\ -\epsilon(a) - \epsilon(a - u_\sigma) & \text{if } k \equiv 2 \pmod 3 \\ -\rho\epsilon(a - u_\sigma) & \text{if } k \equiv 1 \pmod 3 \end{cases} .$$

In the first case, Lemma 4 yields the desired form; in the third case, the result is manifest. Finally, in the second case, Eq. (8) gives

$$-\epsilon(a) - \epsilon(a - u_\sigma) = -\rho^2\epsilon(a + u_{\sigma+1}) .$$

To complete the proof of the theorem, we show by a method of descent that if $\rho = -1, \omega$, or ω^2, then for every n, either

$$\epsilon(n) = \pm\rho^t$$

for some $t \geq 0$, or

$$\epsilon(n) = 0 .$$

Suppose this were false. Then choosing the smallest positive n for which the theorem fails, we need only apply Lemma 5 to arrive at a contradiction. Hence, it suffices to show that n admits a representation

$$n = a + u_{\sigma+k}$$

with $k \geq 2$. We may assume that $n \neq u_t$, since (15) easily implies that

$$\epsilon(u_t) = \frac{\rho\left(1 - \rho^{\left[\frac{t+1}{2}\right]}\right)}{1 - \rho} .$$

which is of the required form for $\rho = -1,\ \omega,$ or ω^2. Taking

$$a = n - u_\nu ,$$

we therefore have $a > 0$. Now

$$a = n - u_\nu < u_{\nu+1} - u_\nu = u_{\nu-1} ,$$

so that

$$\sigma = \nu(a) \le \nu - 2 .$$

Therefore,

$$n = a + u_\nu = a + u_{\sigma+k} ,$$

where $k \ge 2$.

5. APPLICATIONS AND GENERALIZATION

Theorem 1 can be interpreted as a statement about partitions of natural numbers as sums of distinct terms of the sequence $\{u_n\}$ defined by (2).

Letting $A_{k,d}(N)$ denote the number of ways N can be written as a sum

$$N = u_{n_1} + u_{n_2} + \cdots + u_{n_h} ,$$

where $h \equiv d \pmod k$ and

$$n_1 < n_2 < \cdots < n_h ,$$

Theorem 1 asserts that

$$A_{2,0}(N) - A_{2,1}(N) ,$$
$$A_{3,0}(N) - A_{3,1}(N) ,$$
$$A_{3,0}(N) - A_{3,2}(N)$$

are all bounded as N varies over the natural numbers; moreover, if $k \geq 3$, then there exists d such that the difference

$$A_{k,0}(N) - A_{k,d}(N)$$

is not bounded.

Theorem 1 can be proven in the same way for any sequence $\{v_n\}$ such that

$$v_n = v_{n-1} + v_{n-2} ,$$

and can be interpreted as an analogous assertion about partitions of the form

$$N = v_{n_1} + v_{n_2} + \cdots + v_{n_h} .$$

Lemma 5, however, has more precise consequences for the sequence $\{u_n\}$ defined by (2). It is easy to see that $\epsilon(N) = 0$ or ± 1 if $\rho = -1$, and that $\epsilon(N) = 0, \pm 1, \pm\omega$, or $\pm\omega^2$ if $\rho = \omega^2$. The partition-theoretic consequence of this observation is that for each N,

$$\left| A_{2,0}(N) - A_{2,1}(N) \right| \leq 1$$

and

$$\left| A_{3,0}(N) - A_{3,1}(N) \right| + \left| A_{3,1}(N) - A_{3,2}(N) \right| + \left| A_{3,2}(N) - A_{3,0}(N) \right| \leq 1 .$$

NOTE: The truth of Theorem 1 for the special case $\rho = 1$ is a consequence of results found in [4]. The special case $\rho = 1$ is also a consequence of results found in later papers (see [5] and [1]). The interest in series (1) for

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