ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

NOTATION: $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$;
$L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$.

PROPOSED PROBLEMS

B-220 Proposed by Guy A. R. Guillotte, Montreal, P. Q., Canada.

Let p_m be the m^{th} prime. Prove that p_m and p_{m+1} are twin primes (i.e., $p_{m+1} = p_m + 2$) if and only if

$$\sum_{n=1}^{m} (p_{n+1} - p_n) = p_m .$$

B-221 Proposed by R. Garfield, College of Insurance, New York, N. Y.

Prove that

$$\sum_{n=1}^{\infty} (1/F_n L_n) = \sum_{n=1}^{\infty} (1/F_{2n}) .$$

B-222 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Find a formula for K_n where $K_1 = 1$ and

$$K_{n+1} = (K_1 + K_2 + \cdots + K_n) + F_{2n+1} .$$

B-223 Proposed by Edgar Karst, University of Arizona, Tuscon, Arizona.

Find a solution of

$$x^y + (x + 3)^y - (x + 4)^y = u^v + (u + 3)^v - (u + 4)^v$$

in the form

$$x = F_m, \quad y = F_n, \quad u = L_v, \quad \text{and} \quad v = L_s .$$

B-224 Proposed by Lawrence Somer, Champaign, Illinois.

Let m be a fixed positive integer. Prove that no term in the sequence $F_1, F_3, F_5, F_7, \cdots$ is divisible by $4m - 1$.

B-225 Proposed by John Ivie, Berkeley, California.

Let $a_0, \cdots, a_{j-1}$ be constants and let $\{f_n\}$ be a sequence of integers satisfying

$$f_{n+j} = a_{j-1}f_{n+j-1} + a_{j-2}f_{n+j-2} + \cdots + a_0f_n; \quad n = 0, 1, 2, \cdots .$$

Find a necessary and sufficient condition for $\{f_n\}$ to have the property that every integer m is an exact divisor of some f_k.

SOLUTIONS

A SEQUENCE OF MULTIPLES OF 12

B-202 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let $F_1, F_2, \cdots$ be the Fibonacci Sequence $1, 1, 2, 3, 5, 8, \cdots$ with $F_{n+2} = F_{n+1} + F_n$. Let

$$G_n = F_{4n-2} + F_{4n} + F_{4n+2} .$$

(i) Find a recursion formula for the sequence $G_1, G_2, \cdots$.

(ii) Show that each G_n is a multiple of 12.

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

(i) The sequence $\{G_n\}$ satisfies $G_{n+2} = 7G_{n+1} - G_n$ since each of the sequences $\{F_{4n-2}\}$, $\{F_{4n}\}$, and $\{F_{4n+2}\}$ has this recursion relation.

(ii) Since $G_0 = 0$ and $G_1 = 12$, mathematical induction using Part (i) proves that $12 | G_n$ for $n \geq 0$.

Also solved by T. E. Stanley, Gregory Wulczyn, and the Proposer.

A SEQUENCE OF MULTIPLES OF 168

B-203 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Show that $F_{8n-4} + F_{8n} + F_{8n+4}$ is always a multiple of 168.

Solution by T. E. Stanley, City University, London, England.

The following generalizes on B-202 and B-203.

Let

$$E(n,k,r) = F_{kn-r} + F_{kn} + F_{kn+r} .$$

The formulas

$$F_{kn+r} = F_{r-1}F_{kn} + F_r F_{kn+1}$$

$$F_{kn-r} = (-1)^r (F_{r-1} F_{kn} - F_r F_{kn-1})$$

are well known. Thus, if r is even, we have

$$E(n,k,r) = (F_r +)F_{r-1} + 1)F_{kn} .$$

Now F_k divides F_{kn} for each n and so $E(n,k,r)$ is a multiple of

$$(F_r + 2F_{r-1} + 1)F_k$$

for even r. Then $E(n,8,4)$ is a multiple of $(3 + 4 + 1)21 = 168$, which establishes B-203.

Also solved by Gregory Wulczyn and the Proposer.

Editor's Note: Combining thoughts from the solutions of B-202 and B-203, one can show that $F_{kn-2s} + F_{kn} + F_{kn+2s}$ is a multiple of $(L_{2s}+1)F_{kn}$ for $n = 1, 2, 3, \cdots$.

GENERATING FUNCTION FOR F_{2n-1}

B-204 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Show that

(i) $F_1x + F_2x^2 + F_5x^3 + F_7x^4 + \cdots = (x - x^2)/(1 - 3x + x^2)$ for $|x| < (3 - \sqrt{5})/2$.

(ii) $1 + 2x + 3x^2 + 4x^3 + \cdots = 1/(1 - x)^2$ for $|x| < 1$.

(iii) $nF_1 + (n - 1)F_3 + (n - 2)F_5 + \cdots + 2F_{3n-2} + F_{2n-1} = F_{2n+1} - 1$.

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

(i) Let

$$f(x) = (1 - x)/(1 - 3x + x^2)$$

and let its Maclaurin expansion be

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots . \tag{1}$$

Then (1) converges for $|x| < |r|$, where r is the root of $1 - 3x + x^2 = 0$ of least absolute value, i.e., $r = (3 - \sqrt{5})/2$. Multiplying both sides of (1) by $1 - 3x + x^2$ gives us

$$1 - x = (1 - 3x + x^2)c_0 + c_1x + c_2x^2 + \cdots) . \tag{2}$$

Expanding the right side of (2) and equating coefficients of x^m on both sides of (2), leads to

(3) $c_0 = 1, \quad c_1 = 2, \quad c_{n+2} - 3c_{n+1} + c_n = 0 \quad \text{for} \quad n \geq 0 .$

This implies that $c_n = F_{2n+1}$ and Part (i) is proved.

(ii) This follows by term-by-term differentiation of

$$1 + x + x^2 + \cdots = 1/(1 - x), \qquad |x| < 1 .$$

(iii) Let $G_n = nF_1 + (n - 1)F_3 + 2F_{2n-2} + F_{2n-1}$. Then the generating function for the G_n is found by multiplying the series of Parts (i) and (ii) to be

$$1/[(1 - x)(1 - 3x + x^2)] = G_1 + G_2x + G_3x^2 + \cdots .$$

This implies that $G_1 = 1,\ G_2 = 4,\ G_3 = 12,$ and

(4) $$G_{n+3} - 4G_{n+2} + 4G_{n+1} - G_n = 0 .$$

Since $F_{2n+1} - 1$ satisfies the same initial conditions and the same recurrence relation (4) as G_n, Part (iii) is established.

Also solved by the Proposer.

ANOTHER CONVOLUTION FOR F_{2n-1}

B-205 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that

$$(2n - 1)F_1 + (2n - 3)F_3 + (2n - 5)F_5 + \cdots + 3F_{2n-3} + F_{2n-1} = L_{2n} - 2.$$

where L_n is the n^{th} Lucas number (i.e., $L_1 = 1,\ L_2 = 3,\ L_{n+2} = L_{n+1} + L_n$).

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

The solution is similar to that of B-204. Instead of Part (ii) of B-204, one uses

$$1 + 3x + 5x^2 + \cdots = (1 + x)/(1 - x)^2, \quad |x| < 1,$$

which may be obtained by differentiating term-by-term in

$$y + y^3 + y^5 + \cdots = y/(1 - y^2), \quad |x| < 1,$$

and then substituting $y^2 = x$.

A GEOMETRIC SERIES

B-206 Proposed by Guy A. Guillotte, Montreal, Quebec, Canada.

Let $a = (1 + \sqrt{5})/2$ and sum

$$\sum_{n=1}^{\infty} \frac{1}{aF_{n+1} + F_n} .$$

Solution by C. B. A. Peck, State College, Pennsylvania.

From the Fibonacci Quarterly, Vol. 1, No. 3, p. 54,

$$a^{n+1} = aF_{n+1} + F_n .$$

Hence the sum is

$$(1/a^2)[1 - (1/a)] = 1/(a^2 - a) = 1 ,$$

since $a^2 - a - 1 = 0$.

Also solved by Gregory Wulczyn and the Proposer.

ANOTHER GEOMETRIC SERIES

B-207 Proposed by Guy A. Guillotte, Montreal, Quebec, Canada.

Sum

$$\sum_{n=1}^{\infty} \frac{1}{F_n + \sqrt{5}\,F_{n+1} + F_{n+2}} .$$

Solution by C. B. A. Peck, State College, Pennsylvania.

The equation

$$F_n + \sqrt{5}\,F_{n+1} + F_{n+2} = L_{n+1} + \sqrt{5}\,F_{n+1} = 2a^{n+1} ,$$

along with B-206, show that the sum desired here is 1/2.

Also solved by Gregory Wulczyn and the Proposer.

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