Partial Traces on Additive Categories

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Abstract

In this paper, we study partial traces on additive categories. Haghverdi and Scott introduced partially traced symmetric monoidal categories generalizing traced symmetric monoidal categories given by Joyal, Street and Verity. The original example of a partial trace is given in terms of the execution formula on the category of vector spaces and linear functions. Malherbe, Scott and Selinger gave another example of a partial trace on the category of vector spaces, and they observed that we can define these two partial traces on arbitrary additive categories. A natural question is: what kind of partial traces does the category of vector spaces have? We give a (partial) answer to this question. Our main result is: every abelian category has a largest partial trace. Here, "largest" means that every partial trace on the abelian category is obtained by restricting the domain of the largest partial trace. As a corollary, we show that the partial trace given by Malherbe, Scott and Selinger is the largest partial trace on the category of vector spaces.

Keywords: partial trace, execution formula, kernel-image trace, Geometry of Interaction, additive category, abelian category.

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1 Introduction

In computer science, traced monoidal categories [8] and their variants are fundamental algebraic structures in the study of categorical semantics for cyclic computation. For example, a trace operator on a cartesian category is a parametrized fixed point operator [7], and a trace operator on a cocartesian category models iteration. Traced monoidal categories also appear in a categorical framework for Girard's Geometry of Interaction (GoI) [4,1] where trace operators capture interactive communication between automata.

In [5], Haghverdi and Scott introduced partially traced symmetric monoidal categories generalizing the notion of trace in order to give a categorical framework for the original GoI based on vector spaces. Their generalization provides many examples of partially traced symmetric monoidal categories that have not emerged in studies of categorical semantics for lambda calculi. Typically, we need to consider partial traces when the underlying symmetric monoidal category can only model converging iteration process. For example, Haghverdi and Scott showed that the original execution formula

$$\mathrm{Ex}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = A + B(I - D)^{-1}C \tag{1}$$

given in [4] is a partial trace on the symmetric monoidal category of vector spaces with the biproducts as the monoidal products, and they constructed a denotational semantics for linear logic using the execution formula. Roughly speaking, in their work, the execution formula computes the adjacent matrix associated to the normal form of a given proof Π of linear logic. The block matrices A, B, C and D in (1) are adjacent

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matrices corresponding to certain fragments of Π, and when D is nilpotent, (1) is equal to $A+\sum_{n\in\mathbb{N}} BD^nC$, which is the adjacent matrix associated to the normal form of Π. Since the inverse $(I-D)^{-1}$ may not be defined, the execution formula is partially defined. In [12], Malherbe, Scott and Selinger observed that we can generalize construction of the partial trace by Haghverdi and Scott to any additive category, and they also showed that every additive category has another partial trace called the kernel-image trace. A question about these observations is: what kind of partial traces does an additive (or an abelian) category have?

The goal of this paper is to give an answer to this question. We study partial traces on additive categories and abelian categories, and we prove the following results.

- In every additive category, the partial trace given by Haghverdi and Scott is the largest partial trace that satisfies strong naturality. (Theorem 3.3)
- Every abelian category $\mathcal{C}$ has a largest partial trace that is given by restricting the canonical total trace on the category consisting of objects in $\mathcal{C}$ and relations between them. We also show that the largest partial trace is a natural extension of the kernel-image trace on $\mathcal{C}$. (Theorem 4.4 and Theorem 4.5)
- If an abelian category $\mathcal{C}$ is semisimple, then the largest partial trace on $\mathcal{C}$ coincides with the kernel-image trace. (Corollary 4.7)

The second result means that any abelian category has essentially one partial trace; all partial traces on an abelian category are just restrictions of the largest partial trace. It follows from our results that the category of finite dimensional vector spaces equipped with the kernel-image trace is a traced symmetric monoidal subcategory of the category of finite dimensional vector spaces and relations between them. This is an answer to the following question posed in [12]:

> "One question that we did not answer is whether specific partially traced categories can be embedded in totally traced categories in a natural way. For example, the category of finite dimensional vector spaces, with the biproduct $\oplus$ as the tensor, can be equipped with a natural partial trace in several ways. By our proof, it follows that it can be faithfully embedded in a totally traced category. However, we do not know any concrete natural description of such a totally traced category (i.e., other than the free one constructed in our proof)."

Our motivation to study (partial) trace comes from importance of (partial) traces in categorical semantics for recursive computation and GoI, and we are interested in partial traces on additive categories and abelian categories because of use of C*-algebra in the first GoI. However, our results are just answers to purely technical questions on additive categories and abelian categories, and at this point, we do not have any practical application of our results to these research areas. To find practical application of our results is a future work.

The structure of this paper is as follows. In Section 2, we recall the notions of partially traced symmetric monoidal category, additive category and abelian category. We also give some examples and basic properties of these categories. In Section 3, we give examples of partial traces on additive categories and show that the partial trace given by Haghverdi and Scott is the largest partial trace among partial traces satisfying strong naturality. In Section 4, we show that every abelian category has a largest partial trace and when the abelian category is semisimple, the kernel-image trace is the largest partial trace.

2 Categorical preliminaries

2.1 Symmetric monoidal category and symmetric monoidal functor

A *symmetric monoidal category* consists of a category $\mathcal{C}$ equipped with a unit object I and a monoidal product $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and natural isomorphisms $\lambda_X\colon I\otimes X\cong X$, $\rho_X\colon X\otimes I\cong X$, $\alpha_{X,Y,Z}\colon (X\otimes Y)\otimes Z\cong X\otimes(Y\otimes Z)$ and $\sigma_{X,Y}\colon X\otimes Y\cong Y\otimes X$ subject to some coherence conditions (see [11,2] for example). We simply write $(\mathcal{C},I,\otimes)$ or $\mathcal{C}$ for symmetric monoidal categories when we can infer other data from the context. In this paper, we often regard a cartesian category $\mathcal{C}$ as a symmetric monoidal category $(\mathcal{C},1,\times)$ where 1 is the terminal object and $X\times Y$ is the product of X and Y.

Let $(\mathcal{C},I,\otimes)$ and $(\mathcal{D},J,\odot)$ be symmetric monoidal categories. A *symmetric monoidal functor* (F,n,m) from $(\mathcal{C},I,\otimes)$ to $(\mathcal{D},J,\odot)$ consists of a functor $F\colon\mathcal{C}\to\mathcal{D}$ with an arrow $n\colon J\to FI$ and a natural transformation $m_{X,Y}\colon FX\odot FY\to F(X\otimes Y)$ subject to some coherence conditions (see [11,2] for example). A symmetric monoidal functor (F,n,m) is *strong* when n and $m_{X,Y}$ are isomorphisms. We write $F\colon\mathcal{C}\to\mathcal{D}$ to denote a (strong) symmetric monoidal functor when we can infer n and $m_{X,Y}$ from the context.

2.2 Partial trace

We recall the definition of a partial trace introduced by Haghverdi and Scott in [5]. We prepare several notations. Given partially defined expressions $\mathscr{E}$ and $\mathscr{F}$, we write $\mathscr{E}\preceq\mathscr{F}$ when $\mathscr{E}$ is not defined or both $\mathscr{E}$ and $\mathscr{F}$ are defined and they are the same. For example, we have $\sum_{n\in\mathbb{N}} x^n\preceq\frac{1}{1-x}$ for all $x\in[0,\infty)$ because the left hand side is defined if and only if $0 \leq x < 1$, and we have $\sum_{n\in\mathbb{N}} x^n = \frac{1}{1-x}$ for all $0 \leq x < 1$. We write $\mathscr{E} \simeq \mathscr{F}$ when both $\mathscr{E} \preceq \mathscr{F}$ and $\mathscr{F} \preceq \mathscr{E}$ are true.

Definition 2.1 Let $(\mathcal{C}, I, \otimes, \lambda, \rho, \alpha, \sigma)$ be a symmetric monoidal category. A *partial trace* **tr** on $\mathcal{C}$ is a family of partial functions

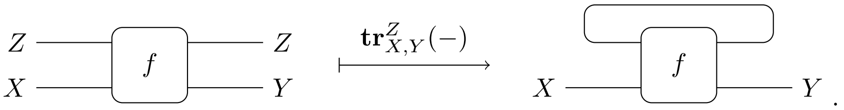
$$\mathbf{tr}^Z_{X,Y} \colon \mathcal{C}(X \otimes Z, Y \otimes Z) \rightharpoonup \mathcal{C}(X, Y) \qquad (X, Y, Z \in \mathcal{C})$$

subject to the following conditions:

- (Naturality) For all $f \colon X \otimes Z \to Y \otimes Z$, $g \colon U \to X$ and $h \colon Y \to V$,
$$g \mathbin{⨾} \mathbf{tr}^Z_{X,Y}(f) \mathbin{⨾} h \preceq \mathbf{tr}^Z_{U,V}((g \otimes Z) \mathbin{⨾} f \mathbin{⨾} (h \otimes Z)).$$
- (Sliding) For all $f \colon X \otimes W \to Y \otimes Z$ and $g \colon Z \to W$,
$$\mathbf{tr}^W_{X,Y}(f \mathbin{⨾} (Y \otimes g)) \simeq \mathbf{tr}^Z_{X,Y}((X \otimes g) \mathbin{⨾} f).$$
- (Superposing) For all $f \colon X \otimes Z \to Y \otimes Z$,
$$W \otimes \mathbf{tr}^Z_{X,Y}(f) \preceq \mathbf{tr}^Z_{W\otimes X, W\otimes Y}(\alpha_{W,X,Z} \mathbin{⨾} (W \otimes f) \mathbin{⨾} \alpha^{-1}_{W,Y,Z}).$$
- (Vanishing I) For all $f \colon X \otimes I \to Y \otimes I$, $\mathbf{tr}^I_{X,Y}(f)$ is defined and is equal to $\rho^{-1}_X \mathbin{⨾} f \mathbin{⨾} \rho_Y$.
- (Vanishing II) For all $f \colon (X \otimes W) \otimes Z \to (Y \otimes W) \otimes Z$, if $\mathbf{tr}^Z_{X\otimes W, Y\otimes W}(f)$ is defined, then
$$\mathbf{tr}^W_{X,Y}(\mathbf{tr}^Z_{X\otimes W, Y\otimes W}(f)) \simeq \mathbf{tr}^{W\otimes Z}_{X,Y}(\alpha^{-1}_{X,W,Z} \mathbin{⨾} f \mathbin{⨾} \alpha_{Y,W,Z}).$$
- (Yanking) For all $X \in \mathcal{C}$, $\mathbf{tr}^X_{X,X}(\sigma_{X,X})$ is defined and is equal to id_X.

A *partially traced symmetric monoidal category* is a symmetric monoidal category equipped with a partial trace. A *total trace* is a partial trace that is totally defined, and a *totally traced symmetric monoidal category* is a symmetric monoidal category equipped with a total trace. We note that the notion of totally traced symmetric monoidal category coincides with the notion of *traced (symmetric) monoidal category* given in [8].

We often present $\mathbf{tr}^Z_{X,Y}(f) \colon X \to Y$ as the following feedback loop:



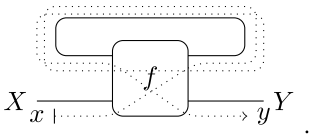
For many totally traced symmetric monoidal categories in computer science, this feedback loop nicely fits in our intuition. As an example, let $(\mathbf{Pfn}, \emptyset, \oplus)$ be the symmetric monoidal category of sets and partial functions whose monoidal product is the disjoint sum

$$X \oplus Y = \{(0, x) : x \in X\} \cup \{(1, y) : y \in Y\}.$$

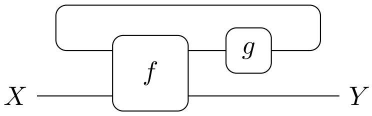
The symmetric monoidal category $(\mathbf{Pfn}, \emptyset, \oplus)$ has a total trace **iter**: for a partial function $f \colon X \oplus Z \rightharpoonup Y \oplus Z$, the trace $\mathbf{iter}^Z_{X,Y}(f) \colon X \rightharpoonup Y$ is given by

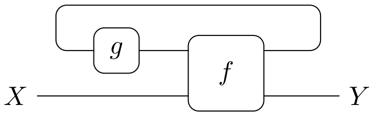
$$\begin{aligned} \mathbf{iter}^Z_{X,Y}(f)(x) = y \iff & \text{there is a finite sequence } z_1, \ldots, z_n \in Z \text{ such that} \\ & f(0, x) = (1, z_1) \text{ and } f(1, z_1) = (1, z_2) \text{ and } \cdots \text{ and } f(1, z_n) = (0, y). \end{aligned}$$

The definition of $\mathbf{iter}^Z_{X,Y}(f)(x)$ means that for $x \in X$, the partial trace $\mathbf{iter}^Z_{X,Y}(f)(x)$ is defined and is equal to $y \in Y$ if and only if there is a finitely many loops like the following dotted line:



This diagrammatic presentation helps us to convince that **iter** is a total trace. For example, the sliding axiom holds because the difference between the following two diagrams





is just how we arrange boxes and wires, and the arrangement has nothing to do with how data flow along wires.

We can construct various partial traces by applying the following theorem to already known partial traces.

Theorem 2.2 ([12, Proposition 3.20]) *Let $(\mathcal{C}, I, \otimes)$ and $(\mathcal{D}, J, \odot)$ be symmetric monoidal categories, and let $\mathbf{tr}$ be a partial trace on $\mathcal{D}$. If there is a faithful strong symmetric monoidal functor $(F, n, m) : (\mathcal{C}, I, \otimes) \to (\mathcal{D}, J, \odot)$, then $\mathcal{C}$ has a partial trace $\underline{\mathbf{tr}}$ given by*

$$\underline{\mathbf{tr}}^{Z}_{X,Y}(f) = \begin{cases} g : X \to Y, & \text{if } \mathbf{tr}^{FZ}_{FX,FY}(m_{X,Z} \fatsemi Ff \fatsemi m^{-1}_{Y,Z}) \text{ is defined and is equal to } Fg, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

This is well-defined since F is faithful.

Example 2.3 Let $(\mathbf{Set}, \emptyset, \oplus)$ be the symmetric monoidal category of sets and functions whose monoidal product is given by the disjoint sum of sets. Since there is a trivial inclusion functor $J : \mathbf{Set} \to \mathbf{Pfn}$, we have a partial trace on **Set** that is given by pulling back **iter** along J. Concretely, the partial trace $\underline{\mathbf{iter}}^{Z}_{X,Y}$ on **Set** is given by

$$\underline{\mathbf{iter}}^{Z}_{X,Y}(f : X \oplus Z \to Y \oplus Z) = \begin{cases} \mathbf{iter}^{Z}_{X,Y}(f), & \text{if } \mathbf{iter}^{Z}_{X,Y}(f) \text{ is totally defined,} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Example 2.4 Let $(\mathbf{Set}, \{0\}, \times)$ be the symmetric monoidal category of sets and functions whose monoidal product is given by the cartesian product of sets, and let $(\mathbf{Rel}, \{0\}, \times)$ be the symmetric monoidal category of sets and relations whose monoidal product is also given by the cartesian product of sets. The symmetric monoidal category $(\mathbf{Rel}, \{0\}, \times)$ has a total trace given by

$$\mathbf{tr}^{Z}_{X,Y}(r) = \{(x, y) \in X \times Y : ((x, z), (y, z)) \in r \text{ for some } z \in Z\}.$$

By pulling back the total trace $\mathbf{tr}$ along the obvious inclusion functor $\mathbf{Set} \to \mathbf{Rel}$, we obtain a partial functor $\underline{\mathbf{tr}}$ on **Set**. To be concrete, given a function $f : X \times Z \to Y \times Z$, the partial trace $\underline{\mathbf{tr}}^{Z}_{X,Y}(f)$ is defined and is equal to a function $g : X \to Y$ if and only if

$$\{(x, y) \in X \times Y \mid f(x, z) = (y, z) \text{ for some } z \in Z\}$$

is the graph relation of g. In the next section, we will generalize this construction to regular categories.

The next example is about existence of the least partial trace.

Example 2.5 For partial traces $\mathbf{tr}_1$ and $\mathbf{tr}_2$ on a symmetric monoidal category $(\mathcal{C}, I, \otimes)$, we write $\mathbf{tr}_1 \leq \mathbf{tr}_2$ when we have $(\mathbf{tr}_1)^{Z}_{X,Y}(f) \preceq (\mathbf{tr}_2)^{Z}_{X,Y}(f)$ for all arrows $f : X \otimes Z \to Y \otimes Z$ in $\mathcal{C}$. This is a partial order on the set of all partial traces on $\mathcal{C}$. With respect to this partial order, every non-empty family of partial traces $\{\mathbf{tr}_\lambda\}_{\lambda \in \Lambda}$ on $\mathcal{C}$ has a greatest lower bound $\mathbf{tr}$ given by

$$\mathbf{tr}^{Z}_{X,Y}(f) \text{ is defined and is equal to } g \iff \text{for all } \lambda \in \Lambda,\ (\mathbf{tr}_\lambda)^{Z}_{X,Y}(f) \text{ is defined and is equal to } g.$$

In particular, if $\mathcal{C}$ has a partial trace, then the meet of all partial traces on $\mathcal{C}$ exists and is the least partial trace on $\mathcal{C}$. We note that the least partial trace on $\mathcal{C}$ is not the everywhere undefined operator because every partial trace must be defined at $\sigma_{X,X}$, and it is not easy to find concrete description of the least partial trace.

2.3 Regular category

In this paper, a partial trace on a regular category plays an important role. In a regular category $\mathcal{C}$, we can consider relations between objects in $\mathcal{C}$, and the category of objects in $\mathcal{C}$ and relations between them has the structure of a totally traced symmetric monoidal category. The partial trace on $\mathcal{C}$ is obtained by pulling back the total trace. Below, we precisely describe this construction. We first recall the definition of a regular category.

Definition 2.6 A *regular category* is a category with finite limits such that

- every arrow $f\colon X \to Y$ has a factorization $f = e \fatsemi m$ with a monomorphism $m\colon Z \to Y$ and a regular epimorphism $e\colon X \to Z$,
- the pullback of a regular epimorphism along any arrow is a regular epimorphism.

Here, an epimorphism $e\colon X \to Y$ is *regular* if and only if e is a coequalizer of a pair of arrows $u, v\colon Z \rightrightarrows X$. Below, we write $e\colon X \twoheadrightarrow Y$ when e is a regular epimorphism from X to Y, and we write $m\colon X \rightarrowtail Y$ when m is a monomorphism from X to Y.

For example, the category **Set** of sets and function is a regular category where the image factorization of a function $f\colon X \to Y$ from a set X to a set Y is:

$$\begin{tikzcd} X \arrow[r, two heads, "f"] & \{y \in Y \mid y = f(x) \text{ for some } x \in X\} \arrow[r, tail] & Y \end{tikzcd} \quad .$$

It is known that regular-epi/mono factorization is unique up-to isomorphism. This means that if an arrow $f\colon X \to Y$ in a category $\mathcal{C}$ has two regular-epi/mono factorizations

$$f = \begin{tikzcd} X \arrow[r, two heads, "e"] & Z \arrow[r, tail, "m"] & Y, \end{tikzcd} \qquad\qquad f = \begin{tikzcd} X \arrow[r, two heads, "e'"] & Z' \arrow[r, tail, "m'"] & Y, \end{tikzcd}$$

then there is an isomorphism $\theta\colon Z \to Z'$ such that $e \fatsemi \theta = e'$ and $\theta \fatsemi m' = m$.

When $\mathcal{C}$ is a regular category, regular-epi/mono factorization enables us to define a totally traced symmetric monoidal category $\mathbf{Rel}(\mathcal{C})$: objects in $\mathbf{Rel}(\mathcal{C})$ are objects in $\mathcal{C}$, and arrows from X to Y are (equivalence classes of) subobjects $r\colon R \rightarrowtail X \times Y$. This is a generalization of the category **Rel** of sets and relations. In fact, we have $\mathbf{Rel}(\mathbf{Set}) = \mathbf{Rel}$. The identity on X is the diagonal arrow $\delta_X\colon X \rightarrowtail X \times X$, and the composition $r * s\colon T \rightarrowtail X \times Z$ of relations $r = \langle r_1, r_2\rangle\colon R \rightarrowtail X \times Y$ and $s = \langle s_1, s_2\rangle\colon S \rightarrowtail Y \times Z$ is defined to be the mono-part of the regular-epi/mono factorization of $\langle p_1 \fatsemi r_1, p_2 \fatsemi s_2\rangle$:

$$\begin{tikzcd} U \arrow[r, two heads] \arrow[rr, bend left, "\langle p_1 \fatsemi r_1, p_2 \fatsemi s_2\rangle"] & T \arrow[r, tail, "r*s"] & X \times Z \end{tikzcd} \quad \text{where} \quad \begin{tikzcd} U \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\text{p.b}"] & S \arrow[d, "s_1"] \\ R \arrow[r, "r_2"'] & Y \end{tikzcd} \quad .$$

Here, $\langle -, - \rangle$ is the tuple of arrows. Well-definedness of the composition of arrows in $\mathbf{Rel}(\mathcal{C})$ follows from uniqueness of regular-epi/mono factorization. The unit object of $\mathbf{Rel}(\mathcal{C})$ is the terminal object $1 \in \mathcal{C}$, and the monoidal product of objects X and Y in $\mathbf{Rel}(\mathcal{C})$ is the product $X \times Y$. The monoidal product of relations $r\colon R \rightarrowtail X \times Y$ and $s\colon S \rightarrowtail Z \times W$ is given by

$$r \otimes s = \begin{tikzcd} R \times S \arrow[r, "r \times s"] & (X \times Y) \times (Z \times W) \arrow[r, "\cong"] & (X \times Z) \times (Y \times W). \end{tikzcd}$$

For a relation $r\colon R \rightarrowtail (X \times Z) \times (Y \times Z)$, the trace $\mathbf{tr}^Z_{X,Y}(r)$ is the mono-part of the regular-epi/mono factorization of the vertical arrow from P to $X \times Y$ in the following diagram:

$$\begin{tikzcd} P \arrow[r, equal] \arrow[d, two heads] & P \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{p.b}"] & & Z \arrow[d, "\delta_Z"] \\ S \arrow[d, tail, "\mathbf{tr}^Z_{X,Y}(r)"'] & R \arrow[rr, "r \fatsemi (\pi'_{X,Z} \times \pi'_{Y,Z})"'] \arrow[d, "r \fatsemi (\pi_{X,Z} \times \pi_{Y,Z})"] & & Z \times Z \\ X \times Y \arrow[r, equal] & X \times Y & & \end{tikzcd} \quad .$$

To clarify intuition of $\mathbf{tr}^Z_{X,Y}(r)$, we concretely describe the above diagram assuming that $\mathcal{C}$ is the category of sets and functions. First, the set R is the graph of r:

$$R = \{((x,z),(y,z')) \in (X \times Z) \times (Y \times Z) \mid ((x,z),(y,z')) \in r\},$$

and two arrows going out from R are projections. The set P is a subset of R given by

$$P = \{((x,z),(y,z)) \in (X \times Z) \times (Y \times Z) \mid ((x,z),(y,z)) \in r\}.$$

Therefore, the image S of the arrow from P to $X \times Y$ is

$$\{(x,y) \in X \times Y \mid ((x,z),(y,z)) \in r \text{ for some } z \in Z\}.$$

We note that we can derive this total trace from the compact closed structure of $\mathbf{Rel}(\mathbf{Set})$, and this is true for arbitrary regular category: the total trace $\mathbf{tr}^Z_{X,Y}$ on $\mathbf{Rel}(\mathcal{C})$ is derived from a *compact closed structure* of $\mathbf{Rel}(\mathcal{C})$ and is called the *canonical trace* in [8]. (We can check that $\mathbf{Rel}(\mathcal{C})$ is a compact closed category by following the idea of the proof of $\mathbf{Rel}$ being compact closed.) For relationship between compact closed categories and total traces, see [8,13]. It is tedious but doable to directly check that $\mathbf{tr}$ is a total trace on $\mathbf{Rel}(\mathcal{C})$.

Now, we can apply Theorem 2.2 to the totally traced symmetric monoidal category $(\mathbf{Rel}(\mathcal{C}), 1, \times, \mathbf{tr})$.

Example 2.7 Let $\mathcal{C}$ be a regular category, and let $H_{\mathcal{C}}$ be a faithful strong symmetric monoidal functor $H_{\mathcal{C}} \colon \mathcal{C} \to \mathbf{Rel}(\mathcal{C})$ given by

$$H_{\mathcal{C}}(X) = X, \qquad H_{\mathcal{C}}(f \colon X \to Y) = \langle \mathrm{id}_X, f\rangle \colon X \rightarrowtail X \times Y.$$

As we have observed, $\mathbf{Rel}(\mathcal{C})$ has a total trace. Therefore, it follows from Theorem 2.2 that $H_{\mathcal{C}}$ gives rise to a partial trace on $\mathcal{C}$. We write $\mathbf{rel}_{\mathcal{C}}$ for this partial trace on $\mathcal{C}$. Concretely, for an arrow $f \colon X \times Z \to Y \times Z$ in $\mathcal{C}$, the partial trace $(\mathbf{rel}_{\mathcal{C}})^Z_{X,Y}(f)$ is defined if and only if there exists $g \colon X \to Y$ in $\mathcal{C}$ such that the mono-part of the regular-epi/mono factorization of $p \fatsemi \langle \pi_{X,Z}, f \fatsemi \pi_{Y,Z}\rangle$ is equal to $\langle \mathrm{id}_X, g\rangle$:

$$\begin{tikzcd}
P \arrow[r, "p"] & X \times Z \arrow[r, "\langle \pi_{X,Z}, f \fatsemi \pi_{Y,Z}\rangle"] & X \times Y \arrow[r, phantom, "="] & P \arrow[r, two heads] & X \arrow[r, tail, "\langle \mathrm{id}_X, g\rangle"] & X \times Y
\end{tikzcd}$$

where $p \colon P \to X \times Z$ is the pullback of δ_Z along $\langle \pi'_{X,Z}, f \fatsemi \pi'_{Y,Z}\rangle$:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\text{p.b}"] & Z \arrow[d, "\delta_Z"] \\
X \times Z \arrow[r, "\langle \pi'_{X,Z}, f \fatsemi \pi'_{Y,Z}\rangle"'] & Z \times Z
\end{tikzcd} ,$$

and when $(\mathbf{rel}_{\mathcal{C}})^Z_{X,Y}(f)$ is defined, $(\mathbf{rel}_{\mathcal{C}})^Z_{X,Y}(f)$ is equal to g.

2.4 Additive categories and abelian categories

We recall the notions of an additive category and an abelian category, and we give their basic properties. For more details, see [11,3].

Definition 2.8 A *pre-additive category* $\mathcal{C}$ is a category such that every hom-set has the structure of an abelian group and the composition is bilinear. We write $0_{X,Y} \colon X \to Y$ for the unit of $\mathcal{C}(X,Y)$ and $f + g$ for the addition of $f, g \colon X \to Y$. An *additive category* is a pre-additive category that has finite coproducts.

The category $\mathbf{Ab}$ of abelian groups and homomorphisms is an additive category. When R is a ring, the category $\mathbf{Mod}_R$ of left R-modules and homomorphisms is also an additive category. In particular, the category $\mathbf{Vect}_K$ of vector spaces over a field K and linear maps is an additive category. We denote the subcategory of $\mathbf{Vect}_K$ consisting of finite dimensional vector spaces by $\mathbf{fdVect}_K$, which is an additive category.

When $\mathcal{C}$ is an additive category, $\mathcal{C}$ has finite products and those coincide with finite coproducts: the initial object 0 is the terminal object, and the coproduct $X \oplus Y$ with

$$\pi_{X,Y} = [\mathrm{id}_X, 0_{Y,X}] \colon X \oplus Y \to X, \qquad \pi'_{X,Y} = [0_{X,Y}, \mathrm{id}_Y] \colon X \oplus Y \to Y \tag{2}$$

is the product of X and Y where $[-,-]$ is the cotuple of arrows. The diagonal arrow $\delta_X\colon X \to X \oplus X$ is given by $\iota_{X,X} + \iota'_{X,X}$ where $\iota_{X,Y}\colon X \to X \oplus Y$ and $\iota'_{X,Y}\colon Y \to X \oplus Y$ are injections. By the definition of projections, we have

$$\iota_{X,Y} \fatsemi \pi_{X,Y} = \mathrm{id}_X \qquad \iota'_{X,Y} \fatsemi \pi'_{X,Y} = \mathrm{id}_Y \qquad \iota_{X,Y} \fatsemi \pi'_{X,Y} = 0_{X,Y} \qquad \iota'_{X,Y} \fatsemi \pi_{X,Y} = 0_{Y,X}$$

and $\pi_{X,Y} \fatsemi \iota_{X,Y} + \pi'_{X,Y} \fatsemi \iota'_{X,Y} = \mathrm{id}_{X\oplus Y}$. The codiagonal arrow $\gamma_X\colon X \oplus X \to X$ is equal to $\pi_{X,X} + \pi'_{X,X}$. The addition $f + g\colon X \to Y$ is equal to $\delta_X \fatsemi (f \oplus g) \fatsemi \gamma_Y$. It follows from coincidence of finite coproducts and finite products that every arrow $f\colon X \oplus Y \to Z \oplus W$ in $\mathcal{C}$ is uniquely decomposed into a matrix of arrows

$$\begin{pmatrix} f_{XZ}\colon X \to Z & f_{XW}\colon X \to W \\ f_{YZ}\colon Y \to Z & f_{YW}\colon Y \to W \end{pmatrix} = \begin{pmatrix} \iota_{X,Y} \fatsemi f \fatsemi \pi_{Z,W} & \iota_{X,Y} \fatsemi f \fatsemi \pi'_{Z,W} \\ \iota'_{X,Y} \fatsemi f \fatsemi \pi_{Z,W} & \iota'_{X,Y} \fatsemi f \fatsemi \pi'_{Z,W} \end{pmatrix}$$

called the matrix decomposition of f. This decomposition is unique because we can recover f as follows:

$$f = \pi_{X,Y} \fatsemi f_{XZ} \fatsemi \iota_{Z,W} + \pi_{X,Y} \fatsemi f_{XW} \fatsemi \iota'_{Z,W} + \pi'_{X,Y} \fatsemi f_{YZ} \fatsemi \iota_{Z,W} + \pi'_{X,Y} \fatsemi f_{YW} \fatsemi \iota'_{Z,W}.$$

The composition of arrows in $\mathcal{C}$ is compatible with matrix multiplication, namely, the matrix decomposition of $f \fatsemi g\colon X \oplus Y \to Z \oplus W \to U \oplus V$ is equal to

$$\begin{pmatrix} f_{XZ} & f_{XW} \\ f_{YZ} & f_{YW} \end{pmatrix} \fatsemi \begin{pmatrix} g_{ZU} & g_{ZV} \\ g_{WU} & g_{WV} \end{pmatrix} = \begin{pmatrix} f_{XZ} \fatsemi g_{ZU} + f_{XW} \fatsemi g_{WU} & f_{XZ} \fatsemi g_{ZV} + f_{XW} \fatsemi g_{WV} \\ f_{YZ} \fatsemi g_{ZU} + f_{YW} \fatsemi g_{WU} & f_{YZ} \fatsemi g_{ZV} + f_{YW} \fatsemi g_{WV} \end{pmatrix}$$

where $(f_{AB}\colon A \to B)_{A,B}$ and $(g_{BC}\colon B \to C)_{B,C}$ are the matrix decompositions of f and g.

Definition 2.9 In a pre-additive category, a *(co)kernel* of $f\colon X \to Y$ is, if it exists, a (co)equalizer of f and $0_{X,Y}$. An *abelian category* is a pre-additive category with finite limits and finite colimits such that every monomorphism is a kernel of some arrow, and every epimorphism is a cokernel of some arrow.

By the definition of abelian categories, every additive category is an abelian category. Categories **Ab**, $\mathbf{Mod}_R$, $\mathbf{Vect}_K$ and $\mathbf{fdVect}_K$ are abelian categories. Let $\mathcal{C}$ be an abelian category. For an arrow $f\colon X \to Y$ in $\mathcal{C}$, we write $Y \twoheadrightarrow \mathrm{coker}(f)$ for the cokernel of $f\colon X \to Y$, and we write $\ker(f) \rightarrowtail X$ for the kernel of $f\colon X \to Y$. Because any epimorphism in $\mathcal{C}$ is a cokernel, epimorphisms in $\mathcal{C}$ are regular epimorphisms. It is known that any arrow $f\colon X \to Y$ in $\mathcal{C}$ has a regular-epi/mono factorization

$$\begin{tikzcd} X \arrow[r, two heads, "e"] & \mathrm{im}(f) \arrow[r, tail, "m"] & Y \end{tikzcd}$$

where m is the kernel of the cokernel $Y \twoheadrightarrow \mathrm{coker}(f)$ and e is the cokernel of the kernel $\ker(f) \rightarrowtail X$. Furthermore, the pullback of an epimorphism along any arrow is an epimorphism. (See [3, Theorem 1.5.5 and Proposition 1.7.6].) Hence, any abelian category is a regular category. Because the definition of abelian category is self-dual, if $\mathcal{C}$ is an abelian category, then $\mathcal{C}^{\mathrm{op}}$ is an abelian category. In particular, $\mathcal{C}^{\mathrm{op}}$ is a regular category.

An object X in a category $\mathcal{C}$ is *projective* when for every epimorphism $f\colon Y \twoheadrightarrow Z$ and for every arrow $g\colon X \to Z$ in $\mathcal{C}$, there is $h\colon X \to Y$ such that $h \fatsemi f = g$. An object $X \in \mathcal{C}$ is *injective* when X is projective in $\mathcal{C}^{\mathrm{op}}$. The following definition is from [9].

Definition 2.10 An abelian category is *semisimple* when one of the following equivalent conditions is true:

- Every object is projective.
- Every object is injective.

The category $\mathbf{Vect}_K$ is semisimple for any field K. More generally, when a ring R is semisimple, i.e., R is a direct sum of its irreducible left R-submodules, then the category $\mathbf{Mod}_R$ of left R-modules is semisimple. For example, finite direct products $K_1 \times \cdots \times K_n$ of fields are semisimple. It follows from Maschke's theorem [10] that given a field K and a finite group G of order n such that the characteristic of K does not divide n, the category $\mathbf{Mod}_{K[G]}$ of left $K[G]$-modules is semisimple where $K[G]$ is the group ring of G over K.

Remark 2.11 In the sequel, we always regard an additive category $\mathcal{C}$ as a symmetric monoidal category $(\mathcal{C}, 0, \oplus)$. As usually, we omit canonical isomorphisms such as $0 \oplus X \cong X \cong X \oplus 0$ and $(X \oplus Y) \oplus Z \cong X \oplus (Y \oplus Z)$ when we can infer them from the context, and we simply write $X \oplus Y \oplus \cdots \oplus Z$ for $((X \oplus Y) \oplus \cdots) \oplus Z$.

Concrete description of fdVect

As an illustration, we explain how the abelian category $\mathbf{fdVect}_K$ looks like. We first note that we can regard each arrow $f\colon K^n \to K^m$ in $\mathbf{fdVect}_K$ as an $n \times m$-matrix. Below, we identify $n \times m$-matrices with arrows from K^n to K^m, and we also identify the monoidal product $K^n \oplus K^m$ with K^{n+m} in the obvious way. Under this identification, the identity on K^n is the $n \times n$ identity matrix I_n, and the composition of arrows in $\mathbf{fdVect}_K$ is the multiplication of matrices. The zero arrow $0_{K^n,K^m}\colon K^n \to K^m$ is the zero matrix, and the addition $f + g\colon K^n \to K^m$ is given by the pointwise manner. The projections $\pi_{K^n,K^m}\colon K^{n+m} \to K^n$ and $\pi'_{K^n,K^m}\colon K^{n+m} \to K^m$ are given by

$$\pi_{K^n,K^m} = \begin{pmatrix} I_n \\ 0_{K^m,K^n} \end{pmatrix}, \qquad \pi'_{K^n,K^m} = \begin{pmatrix} 0_{K^n,K^m} \\ I_m \end{pmatrix},$$

and the injections $\iota_{K^n,K^m}\colon K^n \to K^{n+m}$ and $\iota'_{K^n,K^m}\colon K^m \to K^{n+m}$ are given by

$$\iota_{K^n,K^m} = \begin{pmatrix} I_n & 0_{K^n,K^m} \end{pmatrix}, \qquad \iota'_{K^n,K^m} = \begin{pmatrix} 0_{K^m,K^n} & I_m \end{pmatrix}.$$

It is easy to check the four equations (2) are true for these projections and injections. For example, we have $\iota_{K^n,K^m} \fatsemi \pi_{K^n,K^m} = \mathrm{id}_{K^n}$ because

$$\begin{pmatrix} I_n & 0_{K^n,K^m} \end{pmatrix} \begin{pmatrix} I_n \\ 0_{K^m,K^n} \end{pmatrix} = I_n \fatsemi I_n + 0_{K^n,K^m} \fatsemi 0_{K^m,K^n} = I_n.$$

The matrix decomposition of an arrow $f\colon K^{n+m} \to K^{n'+m'}$ is the partitioning of f into the following 4 blocks:

$$\left(\begin{array}{ccc|ccc} x_{1,1} & \cdots & x_{1,n'} & z_{1,1} & \cdots & z_{1,m'} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n'} & z_{n,1} & \cdots & z_{n,m'} \\ \hline y_{1,1} & \cdots & y_{1,n'} & w_{1,1} & \cdots & w_{1,m'} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & \cdots & y_{m,n'} & w_{m,1} & \cdots & w_{m,m'} \end{array}\right).$$

Given an arrow $f\colon K^n \to K^{n'}$ in $\mathbf{fdVect}_K$, let $u\colon K^n \to K^n$ and $v\colon K^{n'} \to K^{n'}$ be isomorphisms in $\mathbf{fdVect}_K$ such that

$$f = u \fatsemi \begin{pmatrix} I_l & 0_{K^l,K^{n'-l}} \\ 0_{K^{n-l},K^l} & 0_{K^{n-l},K^{n'-l}} \end{pmatrix} \fatsemi v. \tag{3}$$

The natural number l is the rank of f. The regular-epi/mono factorization of f is of the following form:

$$\begin{tikzcd} K^n \arrow[r, two heads, "e"] & K^l \arrow[r, tail, "m"] & K^{n'} \end{tikzcd}$$

where e and m are given by

$$e = u \fatsemi \begin{pmatrix} I_l \\ 0_{K^{n-l},K^l} \end{pmatrix}, \qquad m = \begin{pmatrix} I_l & 0_{K^l,K^{m-l}} \end{pmatrix} \fatsemi v.$$

Because u and v in (3) are not unique, the above decomposition of f is not unique. Still, the decomposition is unique up to isomorphism.

3 Partial traces on additive categories

Because an additive category with a total trace is equivalent to the trivial one-object additive category $\{0\}$ (see Proposition A.1), interesting partial traces on additive categories must be strictly partial. We give some examples of partial traces. The first two examples are from [5,12].

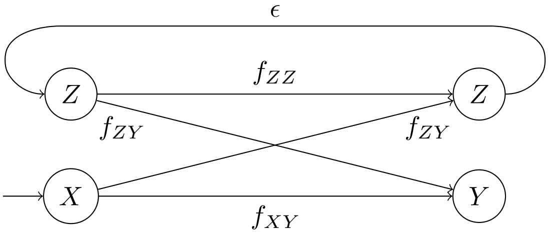
Example 3.1 We define a partial trace $\mathbf{ex}_{\mathcal{C}}$ on an additive category $\mathcal{C}$ by

$$(\mathbf{ex}_{\mathcal{C}})^{Z}_{X,Y}(f) = \begin{cases} f_{XY} + f_{XZ} \fatsemi (\mathrm{id}_Z - f_{ZZ})^{-1} \fatsemi f_{ZY}, & \text{if } \mathrm{id}_Z - f_{ZZ} \text{ is invertible,} \\ \text{undefined,} & \text{otherwise} \end{cases}$$

where $(f_{AB} \colon A \to B)_{A,B}$ is the matrix decomposition of $f \colon X \oplus Z \to Y \oplus Z$. When f_{ZZ} is nilpotent, $\mathrm{id}_Z - f_{ZZ}$ is an isomorphism, and we have

$$(\mathbf{ex}_{\mathcal{C}})^{Z}_{X,Y}(f) = f_{XY} + \sum_{n \in \mathbb{N}} f_{XZ} \fatsemi f^{n}_{ZZ} \fatsemi f_{ZY}. \tag{4}$$

The right hand side is called the *execution formula* [6]. Informally, the result of the execution formula is the accepting language of the following automaton:



where ϵ is the ϵ-transition, and Y is the final state. We note that (4) does not make sense in general as the execution formula subsumes infinite summation.

Example 3.2 We give another partial trace $\mathbf{ki}_{\mathcal{C}}$ called the *kernel-image trace* on an additive category $\mathcal{C}$. For $f \colon X \oplus Z \to Y \oplus Z$, if there are $g \colon X \to Z$ and $h \colon Z \to Y$ such that

$$\begin{tikzcd}
X \arrow[r, "f_{XZ}"] \arrow[dr, "g"'] & Z \arrow[dr, "h"] & \\
& Z \arrow[u, "\mathrm{id}_Z - f_{ZZ}"'] \arrow[r, "f_{ZY}"'] & Y
\end{tikzcd},$$

we define $(\mathbf{ki}_{\mathcal{C}})^{Z}_{X,Y}(f)$ to be $f_{XY} + f_{XZ} \fatsemi h$; otherwise, $(\mathbf{ki}_{\mathcal{C}})^{Z}_{X,Y}(f)$ is undefined. We note that $(\mathbf{ki}_{\mathcal{C}})^{Z}_{X,Y}(f)$ is independent of the choice of g and h. It is easy to see that when $\mathrm{id}_Z - f_{ZZ}$ is invertible, $(\mathbf{ki}_{\mathcal{C}})^{Z}_{X,Y}(f)$ is defined and is equal to $(\mathbf{ex}_{\mathcal{C}})^{Z}_{X,Y}(f)$. Hence, we have $\mathbf{ex}_{\mathcal{C}} \leq \mathbf{ki}_{\mathcal{C}}$. For the definition of the partial order $\leq$ between partial traces, see Example 2.5.

We say that a partial trace $\mathbf{tr}$ on an additive category $\mathcal{C}$ satisfies *strong naturality* when $\mathbf{tr}$ satisfies

$$g \fatsemi \mathbf{tr}^{Z}_{X,Y}(f) \fatsemi h \simeq \mathbf{tr}^{Z}_{U,V}((g \oplus Z) \fatsemi f \fatsemi (h \oplus Z))$$

for all $f \colon X \oplus Z \to Y \oplus Z$, $g \colon U \to X$ and $h \colon Y \to V$ in $\mathcal{C}$. We give a characterization of $\mathbf{ex}_{\mathcal{C}}$.

Theorem 3.3 *Let $\mathcal{C}$ be an additive category. We have:*

- *The partial trace $\mathbf{ex}_{\mathcal{C}}$ satisfies strong naturality.*
- *If a partial trace $\mathbf{tr}$ on $\mathcal{C}$ satisfies strong naturality, then $\mathbf{tr} \leq \mathbf{ex}_{\mathcal{C}}$.*

Hence, $\mathbf{ex}_{\mathcal{C}}$ is the largest partial trace that satisfies strong naturality.

Proof. See Section B. It is easy to check the first claim. For the second claim, the point of the proof is that for a partial trace $\mathbf{tr}$ on $\mathcal{C}$ and for an arrow $f\colon X \oplus Z \to Y \oplus Z$ in $\mathcal{C}$, if

$$\mathbf{tr}^{Z}_{Z,Z}((Z \oplus \delta_Z) \mathbin{⨟} (\sigma_{Z,Z} \oplus f_{ZZ}) \mathbin{⨟} (Z \oplus \gamma_Z))\colon Z \to Z \tag{5}$$

is defined, then it is the inverse of $\mathrm{id}_Z - f_{ZZ}$ where $f_{ZZ}\colon Z \to Z$ is the (Z, Z)th entry of the matrix decomposition of f. We use strong naturality to show that if $\mathbf{tr}^{Z}_{X,Y}(f)$ is defined, then (5) is defined □

We give an example of a partial trace that is smaller than $\mathbf{ex}_{\mathbf{Vect}_K}$. Let $G \subseteq K \setminus \{0\}$ be a subgroup. We define a partial trace $\mathbf{tr}$ on $\mathbf{Vect}_K$ by

$$\mathbf{tr}^{Z}_{X,Y}(f) = \begin{cases} f_{XY} + f_{XZ} \mathbin{⨟} (\mathrm{id}_Z - f_{ZZ})^{-1} \mathbin{⨟} f_{ZY}, & \text{if } \det(\mathrm{id}_Z - f_{ZZ}) \in G, \\ \text{undefined}, & \text{otherwise} \end{cases}$$

where $\{f_{AB}\colon A \to B\}_{A,B}$ is the matrix decomposition of f. We can check that $\mathbf{tr}$ is a partial trace as in the proof of $\mathbf{ex}_{\mathbf{Vect}_K}$ being a partial trace given in [5, Section 2.2]. When G is not equal to $K \setminus \{0\}$, we have $\mathbf{tr} < \mathbf{ex}_{\mathbf{Vect}_K}$.

4 Partial traces on abelian categories

In this section, we study partial traces on abelian categories. When $\mathcal{C}$ is an abelian category, $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are regular categories. Therefore, as we observed in Example 2.7, there are two different constructions of partial traces on $\mathcal{C}$: one is $\mathbf{rel}_{\mathcal{C}}$ that is induced by the strong symmetric monoidal functor $H_{\mathcal{C}}\colon \mathcal{C} \to \mathbf{Rel}(\mathcal{C})$, and the other is $\mathbf{rel}_{\mathcal{C}^{\mathrm{op}}}$ that is induced by the strong symmetric monoidal functor $(H_{\mathcal{C}^{\mathrm{op}}})^{\mathrm{op}}\colon \mathcal{C} \to \mathbf{Rel}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. The goal of this section is to clarify how these partial traces are related and to give their characterizations. Below, we fix an abelian category $\mathcal{C}$.

Proposition 4.1 $\mathbf{rel}_{\mathcal{C}} = \mathbf{rel}_{\mathcal{C}^{\mathrm{op}}}$.

Proof. (Sketch) We define a functor $\Theta\colon \mathbf{Rel}(\mathcal{C}) \to \mathbf{Rel}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ by $\Theta X = X$ and

$$\Theta(\langle r, s\rangle\colon R \rightarrowtail X \oplus Y) = [q, p]\colon Y \oplus X \twoheadrightarrow S \qquad \text{where} \quad \begin{tikzcd} R \arrow[r, "s"] \arrow[d, "r"'] \arrow[dr, phantom, "\text{p.o}"] & Y \arrow[d, "q"] \\ X \arrow[r, "p"'] & S \end{tikzcd} .$$

It is easy to check that $H_{\mathcal{C}} \mathbin{⨟} \Theta$ is equal to $(H_{\mathcal{C}^{\mathrm{op}}})^{\mathrm{op}}$. The functor Θ is a symmetric monoidal isomorphism (Proposition C.2) and preserves total traces. Hence, by the definition of $\mathbf{rel}_{\mathcal{C}}$ and $\mathbf{rel}_{\mathcal{C}^{\mathrm{op}}}$, they are the same.□

We next show that $\mathbf{rel}_{\mathcal{C}}$ is the largest partial trace on $\mathcal{C}$. We prepare two lemmas.

Lemma 4.2 *Let $f\colon X \oplus Z \to Y \oplus Z$ be an arrow in $\mathcal{C}$, let $(f_{AB}\colon A \to B)_{A=X,Z;\, B=Y,Z}$ be the matrix decomposition of f, and let*

$$\begin{tikzcd} Z \arrow[r, two heads, "e"] & W \arrow[r, tail, "m"] & Z \end{tikzcd}$$

be the regular-epi/mono factorization of $\mathrm{id}_Z - f_{ZZ}$. If there are arrows $g\colon X \to W$ and $h\colon W \to Y$ such that the following triangles commute:

$$\begin{tikzcd} X \arrow[r, "f_{XZ}"] \arrow[dr, "g"'] & Z \\ & W \arrow[u, tail, "m"'] \arrow[dr, "h"] & \\ & Z \arrow[u, two heads, "e"] \arrow[r, "f_{ZY}"'] & Y \end{tikzcd} ,$$

then $(\mathbf{rel}_{\mathcal{C}})^{Z}_{X,Y}(f)$ is defined and is equal to $f_{XY} + g \mathbin{⨟} h$.

Proof. We define $p\colon P \twoheadrightarrow X$ to be the pullback of $e\colon Z \twoheadrightarrow W$ along $g\colon X \to W$:

$$\begin{tikzcd}
P \arrow[r, "q"] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\text{p.b}"] & Z \arrow[d, two heads, "e"] \\
X \arrow[r, "g"'] & W
\end{tikzcd} .$$

By the definition of arrows p and q, the tuple $\langle p, q\rangle\colon P \rightarrowtail X \oplus Z$ is the kernel of $\pi_{X,Z} \mathbin{;} g - \pi'_{X,Z} \mathbin{;} e$. Since m is a monomorphism, $\langle p, q\rangle\colon P \rightarrowtail X \oplus Z$ coincides with the kernel of $(\pi_{X,Z} \mathbin{;} g - \pi'_{X,Z} \mathbin{;} e) \mathbin{;} m$, which is equal to $f \mathbin{;} \pi'_{Y,Z} - \pi'_{X,Z}$. Hence,

$$\begin{tikzcd}
P \arrow[r, "q"] \arrow[d, "\langle p,q\rangle"'] & Z \arrow[d, "\delta_Z"] \\
X \oplus Z \arrow[r, "\langle \pi'_{X,Z}, f \mathbin{;} \pi'_{Y,Z}\rangle"'] & Z \oplus Z
\end{tikzcd}$$

is a pullback square. Because p is an epimorphism, the right hand side of

$$\langle p, q\rangle \mathbin{;} \langle \pi_{X,Z}, f \mathbin{;} \pi_{Y,Z}\rangle = p \mathbin{;} \langle \mathrm{id}_X, f_{XY} + g \mathbin{;} h\rangle$$

is the regular-epi/mono factorization of the left hand side of this equation. Hence, $(\mathbf{rel}_{\mathcal{C}})^Z_{X,Y}(f)$ is defined and is equal to $f_{XY} + g \mathbin{;} h$. □

Lemma 4.3 *Let $f\colon X \oplus Z \to Y \oplus Z$ be an arrow in $\mathcal{C}$, let $(f_{AB}\colon A \to B)_{A=X,Z;\, B=Y,Z}$ be the matrix decomposition of f, and let*

$$Z \xrightarrow{e} \mathrel{\mkern-14mu}\rightarrow W \overset{m}{\rightarrowtail} Z$$

be the regular-epi/mono factorization of $\mathrm{id}_Z - f_{ZZ}$. Given a partial trace $\mathbf{tr}$ on $\mathcal{C}$, if $\mathbf{tr}^Z_{X,Y}(f)$ is defined, then there are arrows $g\colon X \to W$ and $h\colon W \to Y$ such that the following triangles commute:

$$\begin{tikzcd}
X \arrow[r, "f_{XZ}"] \arrow[dr, "g"'] & Z & \\
& W \arrow[u, tail, "m"'] \arrow[dr, "h"] & \\
& Z \arrow[u, two heads, "e"] \arrow[r, "f_{ZY}"'] & Y
\end{tikzcd} ,$$

and $\mathbf{tr}^Z_{X,Y}(f)$ is equal to $f_{XY} + g \mathbin{;} h$.

Proof. We first note that $m\colon W \rightarrowtail Z$ is the kernel of the cokernel of $\mathrm{id}_Z - f_{ZZ}$, and $e\colon Z \twoheadrightarrow W$ is the cokernel of the kernel of $\mathrm{id}_Z - f_{ZZ}$. Because $\mathbf{tr}^Z_{X,Y}(f)$ is defined, the right hand side of

$$-f_{XY} + \mathbf{tr}^Z_{X,Y}(f) \simeq \gamma_X \mathbin{;} ((-f_{XY}) \oplus \mathbf{tr}^Z_{X,Y}(f)) \mathbin{;} \delta_Y \preceq \mathbf{tr}^Z_{X,Y}((f_{XZ} \oplus \gamma_Z) \mathbin{;} (\sigma_{Z,Z} \oplus f_{ZZ}) \mathbin{;} (f_{ZY} \oplus \delta_Z))$$

is defined. Therefore, by Lemma B.3,

$$\mathbf{tr}^Z_{X,Z}((f_{XZ} \oplus \delta_Z) \mathbin{;} (\sigma_{Z,Z} \oplus f_{ZZ}) \mathbin{;} ((\mathrm{id}_Z - f_{ZZ}) \oplus \gamma_Z))$$

is defined and is equal to f_{XZ}. Let $c\colon Z \to C$ be the cokernel of $\mathrm{id}_Z - f_{ZZ}$. Then by naturality, we obtain

$$f_{XZ} \mathbin{;} c \preceq \mathbf{tr}^Z_{X,C}((f_{XZ} \oplus f_{ZZ}) \mathbin{;} \gamma_Z \mathbin{;} \iota'_{C,Z}) \simeq \mathbf{tr}^Z_{X,C}(f \mathbin{;} (0_{Y,C} \oplus Z)) \succeq \mathbf{tr}^Z_{X,Y}(f) \mathbin{;} 0_{Y,C} \preceq 0_{X,C}.$$

Hence, there exists a unique $g\colon X \to W$ such that

$$f_{XZ} = X \xrightarrow{g} W \overset{m}{\rightarrowtail} Z .$$

In the same way, we can show that there exists a unique $h\colon W \to Y$ such that

$$f_{ZY} = \begin{tikzcd} Z \arrow[r, two heads, "e"] & W \arrow[r, "h"] & Y \end{tikzcd}.$$

It remains to check that $\mathbf{tr}^{Z}_{X,Y}(f)$ is equal to $f_{XY} + g \fatsemi h$. We define $p\colon P \twoheadrightarrow X$ to be the pullback of $e\colon Z \twoheadrightarrow W$ along $g\colon X \to W$:

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\text{p.b}"] & Z \arrow[d, two heads, "e"] \\ X \arrow[r, "g"'] & W \end{tikzcd} .$$

Because

$$\begin{aligned} p \fatsemi (-f_{XY} + \mathbf{tr}^{Z}_{X,Y}(f)) &\simeq p \fatsemi \gamma_X \fatsemi ((-f_{XY}) \oplus \mathbf{tr}^{Z}_{X,Y}(f)) \fatsemi \delta_Y \\ &\preceq \mathbf{tr}^{Z}_{P,Y}(((p \fatsemi f_{XZ}) \oplus \delta_Z) \fatsemi (\sigma_{Z,Z} \oplus f_{ZZ}) \fatsemi (f_{ZY} \oplus \gamma_Z)) \\ &\simeq \mathbf{tr}^{Z}_{P,Y}(((q \fatsemi (\mathrm{id}_Z - f_{ZZ})) \oplus \delta_Z) \fatsemi (\sigma_{Z,Z} \oplus f_{ZZ}) \fatsemi (f_{ZY} \oplus \gamma_Z)), \end{aligned}$$

it follows from Lemma B.3 that $p \fatsemi (-f_{XY} + \mathbf{tr}^{Z}_{X,Y}(f))$ is equal to $q \fatsemi f_{ZY} = p \fatsemi g \fatsemi h$. Hence, we see that $\mathbf{tr}^{Z}_{X,Y}(f)$ is equal to $f_{XY} + g \fatsemi h$. □

The following theorems are straightforward consequences of Lemma 4.2 and Lemma 4.3. The first one means that $\mathbf{rel}_{\mathcal{C}}$ is the largest partial trace on $\mathcal{C}$, and the second one means that we can define $\mathbf{rel}_{\mathcal{C}}$ without referring to $\mathbf{Rel}(\mathcal{C})$.

Theorem 4.4 *If* $\mathbf{tr}$ *is a partial trace on* $\mathcal{C}$*, then* $\mathbf{tr} \leq \mathbf{rel}_{\mathcal{C}}$*.*

Theorem 4.5 *Let* $f\colon X \oplus Z \to Y \oplus Z$ *be an arrow in* $\mathcal{C}$*, let* $(f_{AB}\colon A \to B)_{A=X,Z;\, B=Y,Z}$ *be the matrix decomposition of* f*, and let*

$$\begin{tikzcd} Z \arrow[r, two heads, "e"] & W \arrow[r, tail, "m"] & Z \end{tikzcd}$$

be the regular-epi/mono factorization of $\mathrm{id}_Z - f_{ZZ}$*. Then we have*

$$(\mathbf{rel}_{\mathcal{C}})^{Z}_{X,Y}(f) = \begin{cases} f_{XY} + h \fatsemi k, & \text{if } f_{XZ} = h \fatsemi m \text{ and } f_{ZY} = e \fatsemi k \text{ for some } h\colon X \to W \text{ and } k\colon W \to Y, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Corollary 4.6 *Any partial trace* $\mathbf{tr}$ *on an abelian category* $\mathcal{C}$ *is* uniform *in the following sense: for all arrows* $f\colon X \oplus Z \to Y \oplus Z$ *and* $g\colon X \oplus W \to Y \oplus W$ *such that* $\mathbf{tr}^{Z}_{X,Y}(f)$ *and* $\mathbf{tr}^{W}_{X,Y}(g)$ *are defined, if there is an arrow* $h\colon Z \to W$ *satisfying*

$$\begin{tikzcd} X \oplus Z \arrow[r, "f"] \arrow[d, "X \oplus h"'] & Y \oplus Z \arrow[d, "Y \oplus h"] \\ X \oplus W \arrow[r, "g"'] & Y \oplus W \end{tikzcd} ,$$

then $\mathbf{tr}^{Z}_{X,Y}(f) = \mathbf{tr}^{W}_{X,Y}(g)$*.*

Proof. We can easily check that $\mathbf{rel}_{\mathcal{C}}$ is uniform in the above sense (See Proposition D.1). Uniformity of arbitrary traces $\mathbf{tr}$ on $\mathcal{C}$ follows from $\mathbf{tr} \leq \mathbf{rel}_{\mathcal{C}}$. □

When $\mathcal{C}$ is a semisimple abelian category, the kernel-image trace $\mathbf{ki}_{\mathcal{C}}$ is equal to the largest partial trace $\mathbf{rel}_{\mathcal{C}}$.

Corollary 4.7 *Let* $\mathcal{C}$ *be a semisimple abelian category. If* $\mathbf{tr}$ *is a partial trace on* $\mathcal{C}$*, then* $\mathbf{tr} \leq \mathbf{ki}_{\mathcal{C}}$*.*

Proof. We show that $\mathbf{ki}_{\mathcal{C}}$ coincides with $\mathbf{rel}_{\mathcal{C}}$. Let $f\colon X \oplus Z \to Y \oplus Z$ be an arrow in $\mathcal{C}$ such that $(\mathbf{rel}_{\mathcal{C}})^{Z}_{X,Y}(f)$ is defined. Let $(f_{AB}\colon A \to B)_{A=X,Z;\, B=Y,Z}$ be the matrix decomposition of f, and let

$$\begin{tikzcd} Z \arrow[r, two heads, "e"] & W \arrow[r, tail, "m"] & Z \end{tikzcd}$$

be the regular-epi/mono factorization of $\mathrm{id}_Z - f_{ZZ}$. By Theorem 4.5, there are $g\colon X \to W$ and $h\colon W \to Y$ such that $f_{XZ} = g \fatsemi m$ and $f_{ZY} = e \fatsemi h$. Because every object is projective and injective, we obtain arrows $g'\colon X \to Z$ and $h'\colon Z \to Y$ such that every triangle in

$$\begin{tikzcd}
X \arrow[r, "f_{XZ}"] \arrow[dr, "g"] \arrow[ddr, "g'"'] & Z \arrow[ddr, "h'"] & \\
& W \arrow[u, tail, "m"'] \arrow[dr, "h"] & \\
& Z \arrow[u, two heads, "e"'] \arrow[r, "f_{ZY}"'] & Y
\end{tikzcd}$$

commutes. Hence, $(\mathbf{ki}_{\mathcal{C}})^{Z}_{X,Y}(f)$ is defined and is equal to $f_{XY} + g' \fatsemi f_{ZY} = (\mathbf{rel}_{\mathcal{C}})^{Z}_{X,Y}(f)$. $\square$

We note that Corollary 4.7 does not hold for $\mathbf{Ab}$, which is not a semisimple abelian category. To see this we show that $\mathbf{ki}_{\mathbf{Ab}} < \mathbf{rel}_{\mathbf{Ab}}$. Let $f\colon \mathbb{Z}_2 \oplus (\mathbb{Z} \oplus \mathbb{Z}_2) \to \mathbb{Z}_2 \oplus (\mathbb{Z} \oplus \mathbb{Z}_2)$ be an arrow in $\mathbf{Ab}$ given by

$$f([x], y, [z]) = ([x], y, [x + y + z])$$

where we write $\mathbb{Z}_2$ for $\mathbb{Z}/2\mathbb{Z}$, and $[n] \in \mathbb{Z}_2$ is the equivalence class of an integer $n \in \mathbb{Z}$. We shall show that $(\mathbf{ki}_{\mathbf{Ab}})^{\mathbb{Z}\oplus\mathbb{Z}_2}_{\mathbb{Z}_2,\mathbb{Z}_2}(f)$ is not defined and that $(\mathbf{rel}_{\mathbf{Ab}})^{\mathbb{Z}\oplus\mathbb{Z}_2}_{\mathbb{Z}_2,\mathbb{Z}_2}(f)$ is defined and is equal to $\mathrm{id}_{\mathbb{Z}_2}$.

- $(\mathbf{ki}_{\mathbf{Ab}})^{\mathbb{Z}\oplus\mathbb{Z}_2}_{\mathbb{Z}_2,\mathbb{Z}_2}(f)$ is not defined because there is no $g\colon \mathbb{Z}_2 \to \mathbb{Z} \oplus \mathbb{Z}_2$ such that

  $$\iota_{\mathbb{Z}_2,\mathbb{Z}\oplus\mathbb{Z}_2} \fatsemi f \fatsemi \pi'_{\mathbb{Z}_2,\mathbb{Z}\oplus\mathbb{Z}_2} = g \fatsemi (\mathrm{id}_{\mathbb{Z}\oplus\mathbb{Z}_2} - \iota'_{\mathbb{Z}_2,\mathbb{Z}\oplus\mathbb{Z}_2} \fatsemi f \fatsemi \pi'_{\mathbb{Z}_2,\mathbb{Z}\oplus\mathbb{Z}_2}).$$

  We note that the left hand side is equal to $\iota'_{\mathbb{Z},\mathbb{Z}_2}$ and $\mathrm{id}_{\mathbb{Z}\oplus\mathbb{Z}_2} - \iota'_{\mathbb{Z}_2,\mathbb{Z}\oplus\mathbb{Z}_2} \fatsemi f \fatsemi \pi'_{\mathbb{Z}_2,\mathbb{Z}\oplus\mathbb{Z}_2}$ maps $(y, [z])$ to $(0, [y])$.
- $(\mathbf{rel}_{\mathbf{Ab}})^{\mathbb{Z}\oplus\mathbb{Z}_2}_{\mathbb{Z}_2,\mathbb{Z}_2}(f)$ is defined and is equal to $\mathrm{id}_{\mathbb{Z}_2}$ because the triangles

  $$\begin{tikzcd}
  \mathbb{Z}_2 \arrow[r, "\iota'_{\mathbb{Z},\mathbb{Z}_2}"] \arrow[dr, "\mathrm{id}_{\mathbb{Z}_2}"'] & \mathbb{Z} \oplus \mathbb{Z}_2 & \\
  & \mathbb{Z}_2 \arrow[u, tail, "\iota'_{\mathbb{Z},\mathbb{Z}_2}"'] \arrow[dr, "0_{\mathbb{Z}_2,\mathbb{Z}_2}"] & \\
  & \mathbb{Z} \oplus \mathbb{Z}_2 \arrow[u, two heads, "e"'] \arrow[r, "0_{\mathbb{Z}\oplus\mathbb{Z}_2,\mathbb{Z}_2}"'] & \mathbb{Z}_2
  \end{tikzcd}$$

  commute where the vertical arrow $e \fatsemi \iota'_{\mathbb{Z},\mathbb{Z}_2}$ is the regular-epi/mono factorization of h. The epimorphism $e\colon \mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$ is given by $e(x, [y]) = [x]$.

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A On a total trace on an additive category

Proposition A.1 *If $\mathcal{C}$ is an additive category with a total trace* $\mathbf{tr}$*, then any $X \in \mathcal{C}$ is isomorphic to* 0.

Proof. Since $0 \to X \to 0$ is the identity, it remains to prove $0_{X,X} = \mathrm{id}_X$. Let $z\colon X \to X$ be

$$\mathbf{tr}^X_{X,X}(\gamma_X \fatsemi \delta_X).$$

Then

$$\begin{aligned}
z &= \mathbf{tr}^X_{X,X}((\delta_X \oplus \delta_X) \fatsemi (X \oplus \sigma_{X,X} \oplus X) \fatsemi (\gamma_X \oplus \gamma_X)) \\
&= \delta_X \fatsemi (X \oplus (\mathbf{tr}^X_{X,X}((X \oplus \delta_X) \fatsemi (\sigma_{X,X} \oplus X) \fatsemi (X \oplus \gamma_X)))) \fatsemi \gamma_X && \text{(naturality, superposing)} \\
&= \mathrm{id}_X + \mathbf{tr}^X_{X,X}((X \oplus \delta_X) \fatsemi (\sigma_{X,X} \oplus X) \fatsemi (X \oplus \gamma_X)) \\
&= \mathrm{id}_X + \mathbf{tr}^X_{X,X}((X \oplus \delta_X) \fatsemi (\sigma_{X,X} \oplus \mathbf{tr}^X_{X,X}(\sigma_{X,X})) \fatsemi (X \oplus \gamma_X)) && \text{(yanking)} \\
&= \mathrm{id}_X + \mathbf{tr}^{X \oplus X}_{X,X}((X \oplus \delta_X \oplus X) \fatsemi (\sigma_{X,X} \oplus \sigma_{X,X}) \fatsemi (X \oplus \gamma_X \oplus X)) && \text{(superposing, naturality, vanishing)} \\
&= \mathrm{id}_X + \mathbf{tr}^{X \oplus X}_{X,X}((X \oplus \sigma_{X,X}) \fatsemi (\gamma_X \oplus X) \fatsemi \sigma_{X,X} \fatsemi (\delta_X \oplus X) \fatsemi (X \oplus \sigma_{X,X})) \\
&= \mathrm{id}_X + \mathbf{tr}^{X \oplus X}_{X,X}((\gamma_X \oplus X) \fatsemi \sigma_{X,X} \fatsemi (\delta_X \oplus X)) && \text{(sliding)} \\
&= \mathrm{id}_X + \mathbf{tr}^X_{X,X}(\gamma_X \fatsemi \mathbf{tr}^X_{X,X}(\sigma_{X,X}) \fatsemi \delta_X) && \text{(vanishing, naturality)} \\
&= \mathrm{id}_X + z. && \text{(yanking)}
\end{aligned}$$

Hence, we obtain $0_{X,X} = \mathrm{id}_X$. □

B A Proof of Theorem 3.3

We divide Theorem 3.3 into Theorem B.2 and Theorem B.4.

Lemma B.1 *Let* $\mathbf{tr}$ *be a partial trace on an additive category. For all $f\colon X \oplus Z \to Y \oplus Z$, we have*

$$W \oplus \mathbf{tr}^Z_{X,Y}(f) \simeq \mathbf{tr}^Z_{W \oplus X, W \oplus Y}(W \oplus f).$$

Proof. If $\mathbf{tr}^Z_{W \oplus X, W \oplus Y}(W \oplus f)$ is defined, then

$$\iota'_{W,X} \fatsemi \mathbf{tr}^Z_{W \oplus X, W \oplus Y}(W \oplus f) \fatsemi \pi'_{W,Y} \preceq \mathbf{tr}^Z_{X,Y}((\iota'_{W,X} \oplus Z) \fatsemi (W \oplus f) \fatsemi (\pi'_{W,Y} \oplus Z)) \simeq \mathbf{tr}^Z_{X,Y}(f).$$

Hence, $\mathbf{tr}^Z_{X,Y}(f)$ is defined. □

Theorem B.2 *Let $\mathcal{C}$ be an additive category. The partial trace $\mathbf{ex}_{\mathcal{C}}$ satisfies strong naturality.*

Proof. For all $f\colon X \oplus Z \to Y \oplus Z$, $g\colon U \to X$ and $h\colon Y \to V$,

$$\begin{aligned}
(\mathbf{ex}_{\mathcal{C}})^Z_{X,Y}(f) \text{ is defined} &\iff \mathrm{id}_Z - \iota'_{X,Z} \fatsemi f \fatsemi \pi'_{Y,Z} \text{ is invertible} \\
&\iff \mathrm{id}_Z - \iota'_{U,Z} \fatsemi ((g \oplus Z) \fatsemi f \fatsemi (h \oplus Z)) \fatsemi \pi'_{V,Z} \text{ is invertible} \\
&\iff (\mathbf{ex}_{\mathcal{C}})^Z_{U,V}((g \oplus Z) \fatsemi f \fatsemi (h \oplus Z)) \text{ is defined.}
\end{aligned}$$

It is easy to see that when $\mathrm{id}_Z - \iota'_{X,Z} \fatsemi f \fatsemi \pi'_{Y,Z}$ is invertible, $(\mathbf{ex}_{\mathcal{C}})^Z_{U,V}((g \oplus Z) \fatsemi f \fatsemi (h \oplus Z))$ is equal to $g \fatsemi (\mathbf{ex}_{\mathcal{C}})^Z_{X,Y}(f) \fatsemi h$. □

In this paper, we often use the next lemma.

Lemma B.3 *Let $\mathcal{C}$ be an additive category, and let* $\mathbf{tr}$ *be a partial trace on $\mathcal{C}$. For all arrows $f\colon X \to Z$, $g\colon Z \to Z$ and $h\colon Z \to Y$ in $\mathcal{C}$, if*

$$\mathbf{tr}^Z_{X,Y}((f \oplus \delta_Z) \fatsemi (\sigma_{Z,Z} \oplus g) \fatsemi (h \oplus \gamma_Z))$$

is defined, then for all $k\colon Z \to W$*,*

$$\mathbf{tr}^{Z}_{X,W}((f \oplus \delta_Z) \fatsemi (\sigma_{Z,Z} \oplus g) \fatsemi (((\mathrm{id}_Z - g) \fatsemi k) \oplus \gamma_Z))$$

is defined and is equal to $f \fatsemi k$*.*

Proof. Let $n_Z\colon Z \to Z$ be $-\mathrm{id}_Z$. We have

$$\begin{aligned}
&\mathbf{tr}^{Z}_{X,W}((f \oplus \delta_Z) \fatsemi (\sigma_{Z,Z} \oplus g) \fatsemi (((\mathrm{id}_Z - g) \fatsemi k) \oplus \gamma_Z)) \\
&\simeq \mathbf{tr}^{Z}_{X,W}((f \oplus \delta_Z) \fatsemi (\sigma_{Z,Z} \oplus (g \fatsemi \delta_Z)) \fatsemi (Z \oplus Z \oplus Z \oplus n_Z) \fatsemi (Z \oplus \sigma_{Z\oplus Z,Z}) \fatsemi ((\gamma_Z \fatsemi k) \oplus \gamma_Z)) \\
&\simeq \mathbf{tr}^{Z}_{X,W}((f \oplus \delta_Z) \fatsemi (\sigma_{Z,Z} \oplus (g \fatsemi \delta_Z)) \fatsemi (Z \oplus Z \oplus Z \oplus \mathbf{tr}^{Z}_{Z,Z}((Z \oplus n_Z) \fatsemi \sigma_{Z,Z})) \fatsemi (Z \oplus \sigma_{Z\oplus Z,Z}) \fatsemi ((\gamma_Z \fatsemi k) \oplus \gamma_Z)) \\
&\qquad \text{(yanking, naturality)} \\
&\simeq \mathbf{tr}^{Z\oplus Z}_{X,W}((f \oplus \delta_Z \oplus Z) \fatsemi (\sigma_{Z,Z} \oplus (g \fatsemi \delta_Z) \oplus n_Z) \fatsemi (Z \oplus \sigma_{Z\oplus Z\oplus Z,Z}) \fatsemi ((\gamma_Z \fatsemi k) \oplus \gamma_Z \oplus Z)) \\
&\qquad \text{(superposing, naturality, vanishing)} \\
&\simeq \mathbf{tr}^{Z\oplus Z}_{X,W}((f \oplus \delta_Z \oplus n_Z) \fatsemi (\sigma_{Z,Z} \oplus g \oplus Z) \fatsemi (Z \oplus \sigma_{Z\oplus Z,Z}) \fatsemi ((\gamma_Z \fatsemi k) \oplus ((Z \oplus \delta_Z) \fatsemi (\gamma_Z \oplus Z)))) \\
&\simeq \mathbf{tr}^{Z\oplus Z}_{X,W}((X \oplus ((Z \oplus \delta_Z) \fatsemi (\gamma_Z \oplus Z))) \fatsemi (f \oplus \delta_Z \oplus n_Z) \fatsemi (\sigma_{Z,Z} \oplus g \oplus Z) \fatsemi (Z \oplus \sigma_{Z\oplus Z,Z}) \fatsemi ((\gamma_Z \fatsemi k) \oplus Z \oplus Z)) \\
&\qquad \text{(sliding)} \\
&\simeq \mathbf{tr}^{Z\oplus Z}_{X,W}((f \oplus \delta_Z \oplus Z) \fatsemi (\sigma_{Z,Z} \oplus \gamma_Z) \fatsemi (k \oplus Z \oplus g)) \\
&\simeq \mathbf{tr}^{Z\oplus Z}_{X,W}((Z \oplus \sigma_{Z,Z}) \fatsemi (f \oplus \delta_Z \oplus Z) \fatsemi (\sigma_{Z,Z} \oplus \gamma_Z) \fatsemi (Z \oplus Z \oplus g) \fatsemi (k \oplus \sigma_{Z,Z})) \qquad \text{(sliding)} \\
&\simeq \mathbf{tr}^{Z}_{X,W}((f \oplus Z) \fatsemi (\delta_Z \oplus Z) \fatsemi (Z \oplus \gamma_Z) \fatsemi (k \oplus g)) \qquad \text{(naturality, superposing, vanishing, yanking)} \\
&\succeq \delta_X \fatsemi ((f \fatsemi k) \oplus \mathbf{tr}^{Z}_{X,W}((f \oplus Z) \fatsemi \gamma_Z \fatsemi g \fatsemi \iota'_{W,Z})) \fatsemi \gamma_W. \qquad \text{(naturality, superposing)}
\end{aligned}$$

Because the left hand side of

$$\mathbf{tr}^{Z}_{X,Y}((f \oplus \delta_Z) \fatsemi (\sigma_{Z,Z} \oplus g) \fatsemi (h \oplus \gamma_Z)) \fatsemi 0_{Y,W} \preceq \mathbf{tr}^{Z}_{X,W}((f \oplus Z) \fatsemi \gamma_Z \fatsemi g \fatsemi \iota'_{W,Z})$$

is defined, $\mathbf{tr}^{Z}_{X,W}((f \oplus Z) \fatsemi \gamma_Z \fatsemi g \fatsemi \iota'_{W,Z})$ is defined and is equal to $0_{X,W}$. Hence,

$$\delta_X \fatsemi ((f \fatsemi k) \oplus \mathbf{tr}^{Z}_{X,W}((f \oplus Z) \fatsemi \gamma_Z \fatsemi g \fatsemi \iota'_{W,Z})) \fatsemi \gamma_W$$

is defined and is equal to $f \fatsemi k$. □

Theorem B.4 *If a partial trace* $\mathbf{tr}$ *on an additive category* $\mathcal{C}$ *satisfies strong naturality, then* $\mathbf{tr}^{Z}_{X,Y}(f) \preceq (\mathbf{ex}_{\mathcal{C}})^{Z}_{X,Y}(f)$ *for all* $f\colon X \oplus Z \to Y \oplus Z$*.*

Proof. Let

$$\begin{pmatrix} f_{XY}\colon X \to Y & f_{XZ}\colon X \to Z \\ f_{ZY}\colon Z \to Y & f_{ZZ}\colon Z \to Z \end{pmatrix}$$

be the matrix decomposition of f. By "strong superposing" (Lemma B.1) and strong naturality, we have

$$\begin{aligned}
\mathbf{tr}^{Z}_{X,Y}(f) &\simeq \delta_X \fatsemi (f_{XY} \oplus \mathbf{tr}^{Z}_{X,Y}(f_{XZ} \oplus \delta_Z) \fatsemi (\sigma_{Z,Z} \oplus f_{ZZ}) \fatsemi (f_{ZY} \oplus \gamma_Z)) \fatsemi \gamma_Y \\
&\simeq f_{XY} + f_{XZ} \fatsemi \mathbf{tr}^{Z}_{Z,Z}((Z \oplus \delta_Z) \fatsemi (\sigma_{Z,Z} \oplus f_{ZZ}) \fatsemi (Z \oplus \gamma_Z)) \fatsemi f_{ZY}.
\end{aligned}$$

Let $g\colon Z \to Z$ be

$$\mathbf{tr}^{Z}_{Z,Z}((Z \oplus \delta_Z) \fatsemi (\sigma_{Z,Z} \oplus f_{ZZ}) \fatsemi (Z \oplus \gamma_Z)).$$

Because $\mathbf{tr}^{Z}_{X,Y}(f)$ is defined, g is also defined. By Lemma B.3, we see that $g \fatsemi (\mathrm{id}_Z - f_{ZZ}) = \mathrm{id}_Z$. By applying the same argument to $\mathcal{C}^{\mathrm{op}}$, we obtain $(\mathrm{id}_Z - f_{ZZ}) \fatsemi g = \mathrm{id}_Z$. Hence, when $\mathbf{tr}^{Z}_{X,Y}(f)$ is defined, $\mathrm{id}_Z - f_{ZZ}$ is invertible and $\mathbf{tr}^{Z}_{X,Y}(f)$ is equal to

$$f_{XY} + f_{XZ} \fatsemi (\mathrm{id}_Z - f_{ZZ})^{-1} \fatsemi f_{ZY},$$

which is $(\mathbf{ex}_{\mathcal{C}})^{Z}_{X,Y}(f)$. □

C $\mathbf{Rel}(\mathcal{C}) \cong \mathbf{Rel}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$

Lemma C.1 *Given a commutative diagram*

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\text{p.o}"] & X \arrow[d, "r"] \arrow[ddr, bend left, "f"] & \\
Y \arrow[r, "s"'] \arrow[drr, bend right, "g"'] & Z \arrow[dr, "u"] & \\
 & & W
\end{tikzcd}$$

in $\mathcal{C}$ such that the inner square $PYZX$ is a pushout square and the outer square $PYWX$ is a pullback square, the unique arrow $u\colon Z \to W$ is a monomorphism.

Proof. By the assumptions, $\langle p, -q\rangle\colon P \to X \oplus Y$ is the kernel of $[f, g]\colon X \oplus Y \to W$, and $[r, s]\colon X \oplus Y \to Z$ is the cokernel of $\langle p, -q\rangle\colon P \to X \oplus Y$. Because the epi-part of the regular-epi/mono factorization of $[f, g]$ is the cokernel of the kernel of $[f, g]$, the arrow u is equal to the mono-part of the regular-epi/mono factorization of $[f, g]$. □

Theorem C.2 Θ *is a strong symmetric monoidal isomorphism from* $\mathbf{Rel}(\mathcal{C})$ *to* $\mathbf{Rel}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ *and preserves total traces.*

Proof. We first check that Θ is a functor. It is easy to see that Θ preserves identities. For relations $f = \langle f_0, f_1\rangle\colon R_0 \rightarrowtail X \oplus Y$ and $g = \langle g_0, g_1\rangle\colon R_1 \rightarrowtail Y \oplus Z$, the composition $f * g\colon R_3 \rightarrowtail X \oplus Z$ is given by the regular-epi/mono factorization of $\langle h_0 \fatsemi f_0, h_1 \fatsemi g_1\rangle$, namely

$$\begin{tikzcd}
\langle h_0 \fatsemi f_0, h_1 \fatsemi g_1\rangle = R_2 \arrow[r, two heads] & R_3 \arrow[r, tail, "f*g"] & X \oplus Z
\end{tikzcd}$$

where h_0 and h_1 are given by the following pullback:

$$\begin{tikzcd}
 & & R_2 \arrow[dl, "h_0"'] \arrow[dr, "h_1"] & & \\
 & R_0 \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & \text{p.b} & R_1 \arrow[dl, "g_0"'] \arrow[dr, "g_1"] & \\
X & & Y & & Z
\end{tikzcd}.$$

Then $\Theta(f * g)$ is given by $[q, p]\colon Z \oplus X \twoheadrightarrow T$ where $p\colon X \to T$ and $q\colon Z \to T$ are given by the pushout of two legs $(f * g) \fatsemi \pi_{X,Z}\colon R_3 \to X$ and $(f * g) \fatsemi \pi'_{X,Z}\colon R_3 \to Z$:

$$\begin{tikzcd}
 & R_2 \arrow[d, two heads] \arrow[ddl, bend right, "h_0 \fatsemi f_0"'] \arrow[ddr, bend left, "h_1 \fatsemi g_1"] & \\
 & R_3 \arrow[dl, "(f*g) \fatsemi \pi_{X,Z}"'] \arrow[dr, "(f*g) \fatsemi \pi'_{X,Z}"] & \\
X \arrow[dr, "p"'] & \text{p.o} & Z \arrow[dl, "q"] \\
 & T &
\end{tikzcd}.$$

Since the inner square is a pushout square, the outer square is a pushout square. Therefore, the outer pushout square is decomposed into the following four pushout squares:

$$\begin{tikzcd}
 & & R_2 \arrow[dl, "h_0"'] \arrow[dr, "h_1"] & & \\
 & R_0 \arrow[dl, "f_0"'] \arrow[dr] & \text{p.o} & R_1 \arrow[dl] \arrow[dr, "g_1"] & \\
X \arrow[dr] & \text{p.o} & U \arrow[dl] \arrow[dr] & \text{p.o} & Z \arrow[dl] \\
 & T_0 \arrow[dr] & \text{p.o} & T_1 \arrow[dl] & \\
 & & T & & .
\end{tikzcd}$$

On the other hand, $\Theta(f) \fatsemi \Theta(g) \colon Z \oplus X \twoheadrightarrow S_3$ is the epi-part of the regular-epi/mono factorization of $[q_1 \fatsemi r_1, p_0 \fatsemi r_0]$, namely

$$[q_1 \fatsemi r_1, p_0 \fatsemi r_0] = Z \oplus X \overset{\Theta(f) \fatsemi \Theta(g)}{\twoheadrightarrow} S_3 \rightarrowtail S_2$$

where p_i, q_i and r_i are given by the following pushouts:

$$\begin{tikzcd}
 & R_0 \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & & R_1 \arrow[dl, "g_0"'] \arrow[dr, "g_1"] & \\
X \arrow[dr, "p_0"'] & \text{p.o} & Y \arrow[dl, "p_1"'] \arrow[dr, "q_0"] & \text{p.o} & Z \arrow[dl, "q_1"] \\
 & S_0 \arrow[dr, "r_0"'] & \text{p.o} & S_1 \arrow[dl, "r_1"] & \\
 & & S_2 & &
\end{tikzcd}$$

By universality of pushouts, we obtain arrows $U \to Y$, $T \to S_2$ and $T_i \to S_i$ for $i = 0, 1$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & R_2 \arrow[dl, "h_0"'] \arrow[dr, "h_1"] & & \\
 & R_0 \arrow[r] \arrow[dl, "f_0"'] \arrow[dr] & U \arrow[d, tail] \arrow[dl] \arrow[dr] & R_1 \arrow[l] \arrow[dl] \arrow[dr, "g_1"] & \\
X \arrow[r] \arrow[dr] & T_0 \arrow[d, tail] \arrow[dr] & Y \arrow[dl] \arrow[dr] & T_1 \arrow[d, tail] \arrow[dl] & Z \arrow[l] \arrow[dl] \\
 & S_0 \arrow[dr] & T \arrow[d, tail] & S_1 \arrow[dl] & \\
 & & S_2 & &
\end{tikzcd}$$

By Lemma C.1, the arrow $U \to Y$ is a monomorphism. Since monomorphisms are stable under pushouts, the vertical arrows $T_0 \to S_0$, $T_1 \to S_1$ and $T \to S_2$ are monic. Because the pair $p \colon X \to T$ and $q \colon Z \to T$ is jointly epi, we obtain the following regular-epi/mono factorization:

$$[q_1 \fatsemi r_1, p_0 \fatsemi r_0] = Z \oplus X \overset{[q,p]}{\twoheadrightarrow} T \rightarrowtail S_2$$

Hence, $\Theta(f \fatsemi g) = [q, p] = \Theta(f) \fatsemi \Theta(g)$.

It remains to check preservation of monoidal products by Θ. Since Θ is identity on objects, $\Theta(0) = 0$. The monoidal product of arrows $f \colon R_0 \rightarrowtail X \oplus Y$ and $g \colon R_1 \rightarrowtail Z \oplus W$ in $\mathbf{Rel}(\mathcal{C})$ is given by

$$\begin{tikzcd} R_0 \oplus R_1 \arrow[r, tail] & (X \oplus Y) \oplus (Z \oplus W) \arrow[r, "\cong"] & (X \oplus Z) \oplus (Y \oplus W)\,. \end{tikzcd}$$

Let

$$\begin{tikzcd}
& R_0 \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & & & & R_1 \arrow[dl, "g_0"'] \arrow[dr, "g_1"] & \\
X \arrow[dr, "p_0"'] & & Y \arrow[dl, "p_1"] & & Z \arrow[dr, "q_0"'] & & W \arrow[dl, "q_1"] \\
& S_0 & & & & S_1 &
\end{tikzcd}$$

be pushout squares. Because

$$\begin{tikzcd}
& R_0 \oplus R_1 \arrow[dl, "f_0 \oplus g_0"'] \arrow[dr, "f_1 \oplus g_1"] & \\
X \oplus Z \arrow[dr, "p_0 \oplus q_0"'] & & Y \oplus W \arrow[dl, "p_1 \oplus q_1"] \\
& S_0 \oplus S_1 &
\end{tikzcd}$$

is a pushout square, $\Theta(f \otimes g)$ is equal to $[p_1 \oplus q_1, p_0 \oplus q_0]$, which is $\Theta(f) \otimes \Theta(g)$. We note that because the traces on $\mathbf{Rel}(\mathcal{C})$ and $\mathbf{Rel}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ are derived from their compact closed structures, it follows from Θ being a strong monoidal functor that Θ preserves traces. By Lemma C.1, we see that Θ is full and faithful. Hence, Θ is an isomorphism of symmetric monoidal categories $\mathbf{Rel}(\mathcal{C})$ and $\mathbf{Rel}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ and preserves total traces. □

D Uniformity of $\mathbf{rel}_{\mathcal{C}}$

Proposition D.1 *For all arrows $f \colon X \oplus Z \to Y \oplus Z$ and $g \colon X \oplus W \to Y \oplus W$ such that $(\mathbf{rel}_{\mathcal{C}})^{Z}_{X,Y}(f)$ and $(\mathbf{rel}_{\mathcal{C}})^{W}_{X,Y}(g)$ are defined, if there is an arrow $h \colon Z \to W$ satisfying*

$$\begin{tikzcd}
X \oplus Z \arrow[r, "f"] \arrow[d, "X \oplus h"'] & Y \oplus Z \arrow[d, "Y \oplus h"] \\
X \oplus W \arrow[r, "g"'] & Y \oplus W
\end{tikzcd}\quad ,$$

then $(\mathbf{rel}_{\mathcal{C}})^{Z}_{X,Y}(f) = (\mathbf{rel}_{\mathcal{C}})^{W}_{X,Y}(g)$.

Proof. Let $(f_{AB} \colon A \to B)_{A=X,Z;\,B=Y,Z}$ and $(g_{AB} \colon A \to B)_{A=X,W;\,B=Y,W}$ be the matrix decompositions of f and g respectively. It follows from the assumption on h that we have

$$f_{XY} = g_{XY}, \qquad f_{XZ} \fatsemi h = g_{XW}, \qquad f_{ZY} = h \fatsemi g_{WY}, \qquad f_{ZZ} \fatsemi h = h \fatsemi g_{WW}.$$

We write

$$\mathrm{id}_Z - f_{ZZ} = \begin{tikzcd} Z \arrow[r, two heads, "e_1"] & V_1 \arrow[r, tail, "m_1"] & Z \end{tikzcd}, \qquad \mathrm{id}_W - g_{WW} = \begin{tikzcd} W \arrow[r, two heads, "e_2"] & V_2 \arrow[r, tail, "m_2"] & W \end{tikzcd}$$

for the regular-epi/mono factorization of $\mathrm{id}_Z - f_{ZZ}$ and the regular-epi/mono factorization of $\mathrm{id}_W - g_{WW}$ respectively. Because e_1 is a regular epimorphism and m_2 is a monomorphism, there is a unique $h' \colon V_1 \to V_2$

such that each square in

$$\begin{tikzcd}
Z \arrow[r, "h"] & W \\
V_1 \arrow[u, tail, "m_1"] \arrow[r, "h'"] & V_2 \arrow[u, tail, "m_2"'] \\
Z \arrow[u, two heads, "e_1"] \arrow[r, "h"'] & W \arrow[u, two heads, "e_2"']
\end{tikzcd}$$

commutes. Since $(\mathbf{rel}_{\mathcal{C}})^{Z}_{X,Y}(f)$ and $(\mathbf{rel}_{\mathcal{C}})^{W}_{X,Y}(g)$ are defined, there are $k_1 \colon X \to V_1$, $k_2 \colon X \to V_2$, $l_1 \colon V_1 \to Y$ and $l_2 \colon V_2 \to Y$ such that

$$f_{XZ} = k_1 \fatsemi m_1, \qquad g_{XW} = k_2 \fatsemi m_2, \qquad f_{ZY} = e_1 \fatsemi l_1, \qquad g_{WY} = e_2 \fatsemi l_2.$$

Because

$$k_2 \fatsemi m_2 = g_{XW} = f_{XZ} \fatsemi h = k_1 \fatsemi m_1 \fatsemi h = k_1 \fatsemi h' \fatsemi m_2,$$

we see that $k_2 = k_1 \fatsemi h'$. Similarly, we obtain $h' \fatsemi l_2 = l_1$. Hence,

$$(\mathbf{rel}_{\mathcal{C}})^{Z}_{X,Y}(f) = f_{XY} + k_1 \fatsemi l_1 = g_{XY} + k_2 \fatsemi l_2 = (\mathbf{rel}_{\mathcal{C}})^{W}_{X,Y}(g).$$

□