CONTEXTS in CONVEX EFFECT ALGEBRAS

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Abstract

A convex effect algebra is an axiomatic structure for quantum mechanics that is based upon physically motivated operations. The basic concept is a two-valued (yes-no) measurement called an effect. The primitive operations on a convex effect algebra (COEA) are a parallel sum and a multiplication by scalars corresponding to an attenuation of effects. The two main examples are a classical COEA of events in a fuzzy probability system and a quantum (or Hilbertian) COEA of positive operators on a Hilbert space. The present work concerns the concept of a context which describes a finest sharp measurement for a physical system. For simplicity, we only consider finite-dimensional contexts. Two important properties of contexts are a spectral condition and a comparability condition. We show that a COEA is isomorphic to a classical COEA if and only if it contains a single spectral context and a COEA is isomorphic to a Hilbertian COEA if and only if its contexts are spectral and comparable. We conjecture that if a COEA has more than one context, then it contains infinitely many. We prove this conjecture for the first few cases.

One of the most important problems in the foundations of physics is to justify the axioms of quantum mechanics on physical grounds. The basic axiom of traditional quantum mechanics is that the pure states are represented by unit vectors in a complex Hilbert space H and the observables are represented by self-adjoint operators on H. But where does H come from? In particular, what do complex numbers have to do with a physical system? What is the physical meaning of the complex inner product $\langle\phi,\psi\rangle$? If self-adjoint operators A and B do not commute, what is the physical meaning of their sum $A+B$ or their product AB? There are many other questions like these. We conclude that the traditional axioms of quantum mechanics are based upon unphysical structures whose basic mathematical operations have no physical meaning.

In this talk we present a mathematical framework for quantum mechanics in which the basic entities and operations have physical significance. The primitive concepts in this framework are states and effects. The states represent initial preparations that describe the condition of the system, while the effects represent yes-no (or zero-one) measurements that probe the system. A state applied to an effect produces the probability that the effect give the value yes (or one). The resulting mathematical structure is a convex effect algebra $\mathcal{E}$. The two mathematical operations on $\mathcal{E}$ are the orthogonal sum $a\oplus b$ and a scalar product λa, $\lambda\in[0,1]\subseteq\mathbb{R}$, both of which having physical interpretations. The sum $a\oplus b$ corresponds to a parallel measurement of a and b while λa corresponds to an attenuation of a by a factor λ.

An *effect-state space* is a triple $(\mathcal{E},\mathcal{S},F)$ where $\mathcal{E}$ and $\mathcal{S}$ are nonempty sets and $F\colon\mathcal{E}\times\mathcal{S}\to[0,1]\subseteq\mathbb{R}$ satisfies:

(ES1) There exist elements $0,1\in\mathcal{E}$ such that $F(0,s)=0$, $F(1,s)=1$ for every $s\in\mathcal{S}$.

(ES2) If $F(a,s)\le F(b,s)$ for every $s\in\mathcal{S}$, then there exists a unique $c\in\mathcal{E}$ such that $F(a,s)+F(c,s)=F(b,s)$ for every $s\in\mathcal{S}$.

(ES3) If $a\in\mathcal{E}$ and $\lambda\in[0,1]\subseteq\mathbb{R}$, then there exists an element $\lambda a\in\mathcal{E}$ such that $F(\lambda a,s)=\lambda F(a,s)$ for every $s\in\mathcal{S}$.

It follows from (ES2) that if $F(a,s)+F(b,s)\le 1$ for every $s\in\mathcal{S}$, then there exists a unique $c\in\mathcal{E}$ such that $F(c,s)=F(a,s)+F(b,s)$ for every $s\in\mathcal{S}$. We then write $a\perp b$ and define $a\oplus b=c$. It also follows from (ES2) that for every $a\in\mathcal{A}$ there exists a unique $a'\in\mathcal{E}$ such that $a\perp a'$ and

$a \oplus a' = 1$. The states on $\mathcal{E}$ have the form $\omega(a) = F(a, s)$ for a fixed $s \in \mathcal{S}$. Notice that $\omega(a \oplus b) = \omega(a) + \omega(b)$ whenever $a \perp b$ and $\omega(1) = 1$. We call $(\mathcal{E}, 0, 1, \oplus)$ a *convex effect algebra*. We define $a \leq b$ if there exists a $c \in \mathcal{E}$ such that $a \oplus c = b$. It can be shown that $(\mathcal{E}, 0, 1, \leq)$ is a bounded poset and $a \perp b$ if and only if $a \leq b'$.

We now present some examples of convex effect algebras. The first example comes from the quantum theory formalism. Let H be a complex Hilbert space and let $\mathcal{E}(H)$ be the set of operators on H that satisfy $0 \leq A \leq I$ where we are using the usual ordering of bounded operators. We take the set of states to be the unit sphere $\mathcal{S}(H)$ in H and for $A \in \mathcal{E}(H)$, $\phi \in \mathcal{S}(H)$ we define $F(A, \phi) = \langle \phi, A\phi \rangle$. Then $(\mathcal{E}(H), \mathcal{S}(H), F)$ is a convex effect algebra which we call a *Hilbertian effect algebra*. In this case we have that $A \oplus B = A + B$ and λA are the usual operator sum and scalar product whenever $A + B \leq I$. Subeffect algebras of $\mathcal{E}(H)$ are called *Hilbertian subeffect algebras*. A von Neumann algebra on H would give an example of a Hilbertian subeffect algebra. Another important example for us comes from fuzzy probability theory. Let $(\Omega, \mathcal{A})$ be a measurable space in which singleton sets are measurable and let $\mathcal{E}(\Omega, \mathcal{A})$ be the set of measurable functions on Ω with values in $[0, 1] \subseteq \mathbb{R}$. The set of states is $\mathcal{S} = \Omega$ with $F(f, \omega) = f(\omega)$. Then $(\mathcal{E}(\Omega, \mathcal{A}), \Omega, F)$ is a convex effect algebra called a *classical effect algebra*. We then have $f \oplus g = f + g$ and λf as the usual function operations whenever $f + g \leq 1$

In the sequel, we let $\mathcal{E}$ be a convex effect algebra (COEA) with set of states $\Omega(\mathcal{E})$. An effect $a \in \mathcal{E}$ is *sharp* if the greatest lower bound $a \wedge a' = 0$. Sharp effects are effects that are precise (unfuzzy). The sharp effects in $\mathcal{E}(\Omega, \mathcal{A})$ are the measurable characteristic functions or equivalently the sets in $\mathcal{A}$. The sharp effects in $\mathcal{E}(H)$ are the projection operators on H. We denote the sharp effects in $\mathcal{E}$ by $S(\mathcal{E})$. An $a \in S(\mathcal{E})$ is *one-dimensional* if $a \neq 0$ and if $b \in \mathcal{E}$ with $b \leq a$ implies that $b = \lambda a$ for some $\lambda \in [0, 1]$. We denote the set of one-dimensional elements by $S_1(\mathcal{E})$. It can be shown that if $a \in S_1(\mathcal{E})$, then there exists a state $\widehat{a} \in \Omega(\mathcal{E})$ such that $\widehat{a}(a) = 1$.

A *context* is a finite set $\{a_1, \ldots, a_n\} \subseteq S_1(\mathcal{E})$ such that

$$a_1 \oplus a_2 \oplus \cdots \oplus a_n = 1 \tag{1}$$

It follows from (1) that $\widehat{a}_i(a_j) = \delta_{ij}$. We interpret a context as a finest sharp measurement. That is, one of the effects a_i must occur and there is no finer sharp measurement. We say that $\mathcal{E}$ is *finite-dimensional* if there exists a context on $\mathcal{E}$. For simplicity, we shall always assume that $\mathcal{E}$ is finite-dimensional. We say that $\mathcal{E}$ is *spectral* if for every $b \in \mathcal{E}$ there exists a context $\{a_1, \ldots, a_n\}$ such that

$$b = \lambda_1 a_1 \oplus \cdots \oplus \lambda_n a_n$$

$\lambda_i \in [0,1]$, $i = 1, \ldots, n$. We now characterize a classical effect algebra $\mathcal{E}(\Omega, \mathcal{A})$. We say that $\mathcal{E}(\Omega, \mathcal{A})$ is *finite* if Ω is finite.

Theorem 1. *A COEA $\mathcal{E}$ is isomorphic to a finite classical effect algebra if and only if $\mathcal{E}$ possesses exactly one context and $\mathcal{E}$ is spectral.*

If $\mathcal{A} = \{a_i \colon i = 1, \ldots, n\}$ is a context we form the set of states $\widehat{\mathcal{A}} = \{\widehat{a}_i, i = 1, \ldots, n\}$. We next construct the complex Hilbert space

$$\mathcal{H}(\mathcal{A}) = \left\{ \sum_{i=1}^{n} \alpha_i \widehat{a}_i \colon \alpha_i \in \mathbb{C} \right\}$$

For $x, y \in \mathcal{H}(\mathcal{A})$ with $x = \sum \alpha_i \widehat{a}_i$, $y = \sum \beta_i \widehat{a}_i$ the inner product is $\langle x, y \rangle = \sum \overline{\alpha}_i \beta_i$. Of course, $\mathcal{H}(\mathcal{A})$ is n-dimensional with orthonormal basis $\widehat{\mathcal{A}}$. We discuss in [1] why we need to take $\mathcal{H}(\mathcal{A})$ to be a complex space instead of a real one. For $b \in \mathcal{E}$ define the linear operator $K_{\mathcal{A}}(b)$ on $\mathcal{H}(\mathcal{A})$ by

$$K_{\mathcal{A}} \left(\sum \alpha_i \widehat{a}_i \right) = \sum \alpha_i \widehat{a}_i(b) \widehat{a}_i$$

The map $K_{\mathcal{A}} \colon \mathcal{E} \to \mathcal{E}\left(\mathcal{H}(\mathcal{A})\right)$ is an affine morphism. That is, $K_{\mathcal{A}}(1) = I$, $K_{\mathcal{A}}(\lambda b) = \lambda K_{\mathcal{A}}(b)$ and $K_{\mathcal{A}}(b_1 \oplus b_2) = K_{\mathcal{A}}(b_1) + K_{\mathcal{A}}(b_2)$. However, $K_{\mathcal{A}}$ need not be injective or surjective and $K_{\mathcal{A}}$ need not preserve sharpness. Moreover, all the $K_{\mathcal{A}}(b)$, $b \in \mathcal{E}$, commute so they do not convey quantum interference. One can say that $K_{\mathcal{A}}$ gives a distorted partial view of $\mathcal{E}$. The reason for this is that we are only employing a single context $\mathcal{A}$. Unlike a classical COEA with only one context, a quantum COEA has many contexts. Each gives a partial view and in order to obtain a total view, they must all be considered.

In order to consider several contexts together, we introduce a method to compare them. A collection of contexts $\Gamma = \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots\}$ is *comparable* if there exist unitary transformations $U_{\mathcal{A}\mathcal{B}} \colon \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{B})$ such that $U_{\mathcal{B}\mathcal{C}} U_{\mathcal{A}\mathcal{B}} = U_{\mathcal{A}\mathcal{C}}$ and

$$\left| \left\langle U_{\mathcal{A}\mathcal{B}} \widehat{a}, \widehat{b} \right\rangle \right|^2 = \widehat{a}(b)$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. We call $\widehat{a}(b)$ the *transition probability from a to b*. It is easy to check that $U_{\mathcal{A}\mathcal{A}} = I$ and $U_{\mathcal{A}\mathcal{B}} = U_{\mathcal{B}\mathcal{A}}^*$. Moreover, defining $\widetilde{U}_{\mathcal{A}\mathcal{B}} \colon \mathcal{E}\left(\mathcal{H}(\mathcal{A})\right) \to \mathcal{E}\left(\mathcal{H}(\mathcal{B})\right)$ by

$$\widetilde{U}_{\mathcal{A}\mathcal{B}} A = U_{\mathcal{A}\mathcal{B}} A U_{\mathcal{B}\mathcal{A}} = U_{\mathcal{A}\mathcal{B}} A U_{\mathcal{A}\mathcal{B}}^*$$

then $\widetilde{U}_{\mathcal{A}\mathcal{B}}$ is an isomorphism.

For a context $\mathcal{A} = \{a_i \colon i = 1, \ldots, n\}$ define the COEA

$$\mathcal{E}(\mathcal{A}) = \{\lambda_1 a_1 \oplus \cdots \oplus \lambda_n a_n \colon \lambda_i \in [0,1], i = 1, \ldots, n\}$$

and for $a = \lambda_1 a_1 \oplus \cdots \lambda_n a_n \in \mathcal{E}(\mathcal{A})$ define $\widetilde{a} \in \mathcal{E}\left(\mathcal{H}(\mathcal{A})\right)$ by

$$\widetilde{a} = \sum_{i=1}^{n} \lambda_i P(\widehat{a}_i)$$

where $P(\widehat{a}_i)$ is the one-dimensional projection onto the subspace generated by the unit vector $\widehat{a}_i$. We say that $\mathcal{E}$ is *strongly comparable* if its set of contexts is comparable and if $b_1 \in \mathcal{E}(\mathcal{A})$, $b_2 \in \mathcal{E}(\mathcal{B})$, $b_1 \oplus b_2 \in \mathcal{E}(\mathcal{C})$ then

$$(b_1 \oplus b_2)^{\sim} = \widetilde{U}_{\mathcal{A}\mathcal{C}} \widetilde{b}_1 + \widetilde{U}_{\mathcal{B}\mathcal{C}} \widetilde{b}_2$$

A slightly different version of the next theorem is proved in [1].

Theorem 2. *A COEA $\mathcal{E}$ is isomorphic to a Hilbertian subeffect algebra if and only if $\mathcal{E}$ is strongly comparable and spectral.*

The isomorphism J in Theorem 2 is constructed as follows. Fix a context $\mathcal{A}$ in $\mathcal{E}$ and for $b \in \mathcal{E}(\mathcal{B})$ define $J \colon \mathcal{E} \to \mathcal{E}\left(\mathcal{H}(\mathcal{A})\right)$ by $J(b) = \widetilde{U}_{\mathcal{B}\mathcal{A}}(\widetilde{b})$.

We have seen in Theorems 1 and 2 that a COEA corresponding to a classical system has a single context while a COEA corresponding to a quantum system may have infinitely many contexts. Are there COES's between these two cases? We conjecture that the answer is no. The next theorem concerns the first few cases.

Theorem 3. *A spectral COEA does not have exactly two or three mutually disjoint contexts.*

References

[1] S. Gudder, Convex and sequential effect algebras, ArXiv: quant-ph, 1802.01265 (2018).