Extended Abstract: Sequential Measurement Characterises Quantum Theory

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This is an extended abstract for [18]. An experimenter doing multiple measurements on a single black-box system will in general see that the order in which measurements are performed is important. For some measurements however, the order of measurement may prove to be unimportant. In this case we call the measurements *compatible*. In this paper we study general physical systems equipped with a notion of sequential measurement that is required to satisfy a few simple and intuitive properties. We show that this forces the system to correspond to a *Euclidean Jordan algebra*. Therefore most of the mathematical structure of quantum theory is recovered just by looking at sequential measurement. Moreover, by additionally requiring composites of systems to satisfy *local tomography* we recover the complete standard description of quantum theory where systems are described by C*-algebras and they compose via the tensor product. In other words, we show that quantum theory is the unique locally tomographic theory allowing sequential measurement.

A measurement done on a physical system will generally disturb it. As a result, the order in which measurements are performed on a fixed system will be important. We will show how this simple observation can be used to derive the mathematical content of quantum theory.

Suppose we have access to some black-box state on which we are allowed to perform measurements that have a binary output. We will refer to one of the outcomes as the measurement being 'successful'. An example would be shining a laser on a particle and seeing whether a photon is emitted. We will refer to a binary measurement as an *effect*. Suppose that we perform an initial measurement a and that afterwards we perform a potentially different measurement b. We define the *sequential product* of a and b to be the effect that is implemented by first measuring a and then measuring b. The sequential product, that we will denote by $a \& b$ is considered successful when a and b are both successful so that we can read the expression $a \& b$ as "we measure *a and then* we measure b."

Since both the measurements can influence the system in some non-trivial way, we wouldn't expect the outcome statistics of $a \& b$ to be the same as those of $b \& a$, the measurement that is implemented by reversing the order of measurement. For some measurements however, the order might not be important. When this is the case we will call the measurements *compatible*. Following Gudder and Greechie [6] we will argue that when compatible measurements are considered, the sequential product should act in a 'classical' way (see Definition 1). Given these considerations pertaining compatible measurements we will show that the set of measurements and states of the black-box system is a *Euclidean Jordan algebra*, a type of space originally introduced by Jordan, von Neumann and Wigner as an algebraic generalisation of quantum theory [12, 13]. An experimenter who does not know the laws of quantum theory might therefore already derive a lot of its structure simply by modelling how he would expect sequential measurement to behave.

Physical system do of course not exist in a vaccuum and any complete description of a physical theory should therefore describe how systems compose together into larger systems. We call a composition

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of systems *locally tomographic* when a joint state is completely determined by the statistics of measurements on its component systems. We will show that any physical theory that is locally tomographic and that has a sequential measurement satisfying our conditions, must be a sub-theory of quantum theory: that systems correspond to (finite-dimensional) C*-algebras and systems compose via the tensor product. This result gives a clear underpinning of the mathematical content of quantum theory via some simple arguments concerning sequential measurement and composites of systems.

The framework we will use to derive our results is that of *generalised probabilistic theories* [3]. In this framework we can take probabilistic combinations of states and effects by tossing a biased coin and preparing a state or effect based on the outcome. So let Ω be the set of states a given system can be in and let E be the set of effects we can measure on the system. The possibility of taking probabilistic combinations translates to the requirement of the set of states Ω and the set of effects E to be convex sets. We can always view the set of effects E as the unit interval of some ordered vector space V such that $E = [0,1]_V = \{a \in V \; ; \; 0 \leq a \leq 1\}$ [3]. Here 1 is the effect that is always successful ($\omega(1) = 1$ for all states $\omega \in \Omega$), and 0 is the effect that is never successful: ($\omega(0) = 0$). For a given effect a we can also consider its *negation* $a^\perp := 1 - a$ that is defined by a measurement of a and then negating its outcome.

We require our ordered vector space V to be an *order unit space*. This is a space satisfying the following implication: $\forall n \in N : a \leq b + \frac{1}{n} \implies a \leq b$. Intuitively it means that there don't exist any infinitesimal effects. An order unit space has a natural choice of norm [1]. When we refer to continuity it should be understood to refer to the topology induced by this norm.

We can now state what we will consider to be a well-behaved sequential product:

Definition 1. Let V be an order unit space and let $E = [0,1]_V$. Given a function $\& : E \times E \to E$ we write $a \,|\, b$ when $a \,\&\, b = b \,\&\, a$, i.e. when a and b are compatible. We call & a *sequential product* when it satisfies the following conditions for all $a, b, c \in E$.

- Additivity and continuity: $a \,\&\, (b + c) = a \,\&\, b + a \,\&\, c$ and the map $b \mapsto a \,\&\, b$ is continuous.
- Unitality and orthogonality: $1 \,\&\, a = a$ and if $a \,\&\, b = 0$ then also $b \,\&\, a = 0$.
- Compatible associativity: If $a \,|\, b$ then $a \,\&\, (b \,\&\, c) = (a \,\&\, b) \,\&\, c$.
- Classicality: If $a \,|\, b$ then $a \,|\, b^\perp$, and if additionally $a \,|\, c$ then $a \,|\, (b + c)$ and $a \,|\, (b \,\&\, c)$.

Note that except for the condition of continuity these properties are exactly the ones required in a *sequential effect algebra* as defined by Gudder and Greechie [6]. It is worth remarking that this product is in general not bilinear, commutative or associative. For a conceptual explanation of the properties in Definition 1 we refer to the main paper [18].

Letting $V = B(H)^{\mathrm{sa}}$ be the set of Hermitian operators on a complex Hilbert space H, we get the usual description of quantum theory. Given two effects a and b, their sequential product is defined as $a \,\&\, b := \sqrt{a} b \sqrt{a}$ (and we have $a \,|\, b$ if and only if $ab = ba$ [8]). It is not immediately obvious why this map should indeed take on the role of the sequential product. To remedy this a few sets of reasonable postulates that characterise it have been found [7, 16, 19]. The properties given in Definition 1 are *not* enough to fully determine the standard sequential product [15].

Instead of letting the order unit space be $B(H)^{\mathrm{sa}}$ where H is a *complex* Hilbert space, we can also let it be a *real* or even *quaternionic* Hilbert space and the product $a \,\&\, b := \sqrt{a} b \sqrt{a}$ would still define a sequential product. This observation leads to the first main theorem:

Theorem 2. The only finite-dimensional order unit spaces with sequential product, being of rank at least 4 and having no non-trivial classical effects are of the form $B(H)^{\mathrm{sa}}$ where H is a real, complex or quaternionic Hilbert space.

Here the *rank* of a space is equal to the maximal amount of orthogonal *sharp* effects. An effect p is sharp when it satisfies $p \,\&\, p = p$. An effect is *classical* when it is compatible with all other effects. When a space contains non-trivial classical effects it can be split into a direct sum of smaller systems. The conditions regarding the rank of the space and the non-existence of classical effects are needed to exclude a few other possibilities. Without these requirements the spaces we get are the *Euclidean Jordan algebras*. These algebras turn out be the most general kind of space where a sequential product can be defined.

Theorem 3. A finite-dimensional order unit space with a sequential product must be a Euclidean Jordan algebra.

Note: The converse that every Euclidean Jordan algebra allows a sequential product is shown in [19].

The above results only concern single systems without any notion of interaction between different systems. When we do consider possible composites of systems, the type of spaces that are allowed is even more restricted. Following other reconstructions of quantum theory [5, 14, 9, 11] we require our composites to be *locally tomographic*. This says that a state on a composite system is completely determined by local measurements on the subsystems. This leads us to our main theorem.

Theorem 4. Any locally tomographic generalised probabilistic theory that allows sequential measurement must be (a sub-theory of) quantum theory.

A sub-theory of quantum theory is one where the systems can be identified with (finite-dimensional) C*-algebras, and the composite is given by the standard tensor product. The most general type of map that sends effect to effects (or equivalently states to states) are then the completely positive maps.

This theorem can be seen as an intuitive reconstruction of quantum theory. It is worthy of note that whereas other reconstructions of quantum theory generally use interactions between different axioms that refer to concepts such as purity, spectrality or the existence of certain dynamics [2, 10, 4, 17], this reconstruction is done by fully exploiting the weak algebraic structure supplied by the sequential product, and not much else. It is also worth noting that with a few additional technical conditions Theorem 2 also holds for spaces of infinite-rank.

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