Almost quantum correlations violate the no-restriction hypothesis

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To identify which principles characterize quantum correlations, it is essential to understand in which sense this set of correlations differs from that of almost quantum correlations. We solve this problem by invoking the so-called no-restriction hypothesis, an explicit and natural axiom in many reconstructions of quantum theory. The no-restriction hypothesis basically states that the set of possible measurements is the dual of the set of states. We prove that, contrary to quantum correlations, no generalised probabilistic theory satisfying the no-restriction hypothesis is able to reproduce the set of almost quantum correlations. Therefore, any theory whose correlations are exactly, or very close to, the almost quantum correlations necessarily requires a rule limiting the possible measurements. Our results suggest that the no-restriction hypothesis may play a fundamental role in singling out the set of quantum correlations among other non-signalling ones.

One of the most pronounced phenomena to reveal that Nature is not classical is Bell nonlocality. Indeed, quantum realisations of the so-called 'Bell experiment' have confirmed that Bell inequality violations can be observed. The study of quantum nonlocality has hence become an active field of research aimed, on the one hand, at exploiting these correlations with no classical analogue and, on the other, at understanding the counter-intuitive features of quantum theory, which represents our most accurate description of nature to date.

Part of this research has focused on deriving the correlations produced by quantum mechanics solely from physical principles, i.e. without making reference to the underlying mathematical structure of Hilbert spaces, vectors, self-adjoint operators and so forth. This program was initiated by Popescu and Rohrlich [1] (see also [2]), who showed that relativistic considerations (no-signalling faster than light) alone are not enough to single out the quantum set, and in fact allow two spatially separated parties to generate correlations impossible to approximate in quantum theory.

More than a decade later, Van Dam showed that the existence of some supra-quantum correlations compatible with special relativity could have implausible consequences from an information processing point of view [3]. Since then, remarkable progress has been made in identifying physical principles that constrain non-signalling correlations [4, 5, 6, 7, 8]. However, all these attempts turned out to be insufficient in singling out quantum correlations, since (with the possible exception of [6]), they proved to be satisfied by a set of correlations slightly larger than the quantum one, called *almost quantum* (AQ)

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[9]. This opens the question of whether there is a fundamental limitation to this research program, or if it is the case that almost quantum correlations could really be the correlations allowed by a theory alternative to quantum mechanics.

In this work we discuss the properties that a physical theory should have in order to predict AQ correlations. For this we work within the framework of generalized probabilistic theories (GPTs). We find that should such a theory exist, it will not satisfy the no-restriction hypothesis, in the sense that the measurements allowed on the systems should be further constrained. Indeed, the no-restriction hypothesis imposes that any mathematically well defined measurement in the theory should be physically allowed. Depending on the interpretation, our results then provide a means to argue that AQ correlations may not be physical after all, or on the contrary, highlight how restrictive the no-restriction hypothesis is.

Bell scenarios and the almost quantum set

In a Bell scenario, a set of distant parties perform space-like separated actions on their share of a system. Each party can choose from different measurements to perform on their subsystem, obtaining in each case one out of many possible outcomes. For the purpose of this short abstract, let us focus initially on the case of two parties, Alice and Bob. We use x (y) to denote Alice's (Bob's) choice of measurement setting and a (b) to denote the obtained outcome. After performing measurements on many independent copies of the system, the parties can estimate the conditional probability distribution (also called behaviour) $p(ab|xy)$. These correlations are the objects we aim to characterise.

The correlations $p(ab|xy)$ that the parties can observe are limited by the physical theory modelling the behaviour of their experimental apparatuses. Quantum theories allow correlations of the form $p(ab|xy) = \langle\Psi|\, M^A_{a|x} \otimes M^B_{b|y}\, |\Psi\rangle$, where $\Psi \in \mathscr{H}_A \otimes \mathscr{H}_B$ is a normalized vector; $\mathscr{H}_A$ and $\mathscr{H}_B$ are arbitrary Hilbert spaces and $\{M^{(A)}_{a|x}\}_a$ is a positive operator valued measure (POVM) acting on $\mathscr{H}_A$ for all x, and similarly for Bob.

The set of AQ correlations, denoted by $\tilde{\mathscr{Q}}$, can be mathematically defined in many equivalent ways. Within the language of quantum theory, a behaviour $p(ab|xy)$ has an AQ realisation if there exists a Hilbert space $\mathscr{H}$, projective measurements $\{N^{(A)}_{a|x}\}_a$ and $\{N^{(B)}_{b|y}\}_b$ acting on $\mathscr{H}$, and a quantum state $|\Psi\rangle$ such that (i) $p(ab|xy) = \langle\Psi|\, N^A_{a|x} N^B_{b|y}\, |\Psi\rangle$, and (ii) $N^A_{a|x} N^B_{b|y}\, |\Psi\rangle = N^B_{b|y} N^A_{a|x}\, |\Psi\rangle$.

As shown in [9], all quantum correlations belong to the AQ set, while in turn there exist instances of AQ correlations which do not admit a quantum realization. The set of AQ correlations, moreover, have a further property that the quantum one lacks: contrary to the quantum case, the problems of testing membership in the AQ set as well as computing maximal violations of Bell inequalities therein can be solved efficiently via semidefinite programming (SDP) [10].

GPT: the essentials

In order to explore the properties of a hypothetical theory that describes AQ correlations, we make use of the formalism of generalised probabilistic theories (GPTs) [11, 12], also referred to as the convex operational framework. The three main concepts that we make use of are the following:

- In a GPT, each system is described by a *state* Ψ which is fully specified by the vector of probabilities for the outcomes of all measurements (or a tomographically complete subset of them) that can be performed on it. The set of possible states $\mathscr{S}$ of a given system type is always convex.
- An *effect* $\mathbf{e}$ is a linear functional on $\mathscr{S}$ that maps each state onto a probability, i.e. a real number between 0 and 1. The set of linear functionals with that property form the set $\mathscr{S}^*$, the dual of $\mathscr{S}$.

- Any d-outcome *measurement* can be specified by a collection of d effects $\mathbf{e}_j$ such that $\sum_{j=1}^{d} \mathbf{e}_j(\Psi) = 1$ for all valid states Ψ. The probability of obtaining outcome a when that measurement is performed on Ψ is given by $\mathbf{e}_a(\Psi)$.

A theory in which all elements of $\mathscr{S}^*$ *are* allowed effects is called 'dual'. The property of duality is often assumed as a starting point in derivations of quantum theory [13, 12, 14, 15, 16], and is usually referred to as *the no-restriction hypothesis* [17]. This hypothesis can be shown to follow from the existence of orthonormal bases for the state space and the equivalence of such bases under reversible transformations [18].

Any hypothetical AQ theory violates the no-restriction hypothesis

The main result of our paper is to show that any GPT that makes accessible the set of AQ correlations cannot satisfy the no-restriction hypothesis.

The key point in the proof is to identify within the dual set of states for two parties, $\mathscr{S}_{AB}^*$, two-party functionals $\{U_i\}$ that display an inconsistent property. This problematic property is the following:

- The functionals $\{U_i\}$ are well defined two-party functionals, namely $U_i(\Psi_{AB}) \in [0,1]$ for all i, and for all joint states $\Psi_{AB} \in \mathscr{S}_{AB}$ for Alice and Bob.
- When Alice and Bob act as a single party in a bipartite Bell scenario, where Charlie is the second party, should they choose their local measurements from $\{U_i\}$, they would be able to be correlated with Charlie in a way that goes beyond the strength of AQ correlations.

Hence, should the GPT satisfy the no-restriction hypothesis, then these functionals $\{U_i\}$ would be allowed effects in the theory, which in turn will now be not consistent with itself (i.e. the GPT will not be self-consistent under composition).

The novel idea to find joint effects with the above mentioned inconsistency property, relies on realising that Bell functionals can be renormalised to become AQ allowed functionals for joint systems, which the no-restriction hypothesis then promotes to be valid AQ effects.

Conclusions

In this paper we have shown that any theory that predicts almost quantum correlations must violate the no-restriction hypothesis. This is the *first* time that a fundamental principle satisfied by quantum theory proves to be violated by almost quantum. Our work then reaches a milestone towards solving the outstanding problem of characterising quantum behaviours, which had been stalled for half a decade.

Besides its fundamental relevance, from a more general perspective, our formalism provides a way to test the validity of the no-restriction hypothesis in a device-independent way. Moreover, it further opens two main research questions that deserve further investigation. First, it would be interesting to identify GPT theories predicting AQ correlations and the constraints they impose on the set of measurements. Second, one may wonder which sets of correlations are compatible with the no-restriction hypothesis. General non-signalling and quantum correlations are examples of those sets. How to identify other such sets is left for future work. In particular, leaving aside general non-signalling correlations, is there any set of correlations that is strictly larger than the quantum set and does not violate the no-restriction hypothesis?

A technical version of the results may be found in this arxiv preprint, which is being under review in a different journal.

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