Small violations of Bell inequalities for multipartite pure random states (Extended Abstract)

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I. SMALL PROBABILITIES OF HIGH VIOLATIONS

It is already known [1–3] that if either the local dimension d of each quantum system, or the number N of parts, is sufficiently large, then typical quantum states are highly entangled. Should we expect, then, that typically there will be at least one Bell inequality that is greatly violated? Our main objective in the present work is, then, to approach the following question:

given a typical pure state $|\psi\rangle$ composed by N $d-$dimensional quantum systems, what should one expect for its largest possible violation over all relevant Bell inequalities in a given scenario $\Gamma = (N,m,v)$?

In our framework, optimal violations (if any) of Bell inequalities exhibited by a quantum state $|\psi\rangle \in (\mathbb{C}^d)^{\otimes N}$ are given by the functional:

$$V_{\text{opt}}(|\psi\rangle) := \sup_{\substack{A\in\mathscr{A}\\ T\in\mathscr{T}_b}} |Q(|\psi\rangle, T, A)|, \tag{1}$$

where the supremum is taken over all quantum implementations of all Bell inequalities whose coefficients are *uniformly bounded by b*.

Now, if $|\psi\rangle$ is a point of a sample space, since V_{opt} is a function of $|\psi\rangle$ we can consider it as a random variable. Formally, what we would like to estimate is the distribution function of such variable, that is:

$$\mathbb{P}(V_{\text{opt}} > c). \tag{2}$$

It is known [4, 5] that any entangled N-partite pure state violate *some* Bell inequality. Since these states have full measure on the sphere of pure states, its is clear then that $\mathbb{P}(V_{\text{opt}} > c) = 1$ for $c \leq 1$. For arbitrary $c > 1$ we have our main result:

Theorem 1. *Given $N, d \geq 2$ integers. Let $|\psi\rangle \in \left(\mathbb{C}^d\right)^{\otimes N}$ be a unit vector distributed according to the uniform measure in the sphere S_{2d^N-1} of $\left(\mathbb{C}^d\right)^{\otimes N}$, then:*

$$\mathbb{P}(V_{\text{opt}} > c) \leq 4\left[\frac{8bN(mv)^N d^2}{\delta} + 2\right]^{mvNd^2+(mv)^N} \times e^{-\left(\frac{2d^N(c-\delta-1)^2}{36\pi^2(2m-1)^{2N-2}}\right)}, \tag{3}$$

for any $b, \delta > 0$, $c > \delta + 1$.

This theorem allows us to answer the question posed at the beginning of this section negatively. A typical state $|\psi\rangle$ composed by N d-dimensional quantum systems, with N and/or d large enough, *does not* exhibit a significant degree of violation for any Bell inequality (with uniformly bounded coefficients).

On the one hand, if the local dimension d of each subsystem satisfies

$$d > mv(2m-1)^2, \tag{4}$$

and if in addition the uniform bound b is not too large, *e.g.* if

$$b = \mathscr{O}\left((mn)^N\right), \tag{5}$$

then the fifth term in brackets dominates all other terms. So, we are left with:

$$\mathbb{P}(V_{\text{opt}} > c) \to 0 \tag{6}$$

super-exponentially fast as $N \to \infty$. Consequently:

if the local dimension d of a N-partite quantum system satisfies $d > mv(2m-1)^2$, then, for large N, the vast majority of pure states will not violate any Bell inequality with bounded coefficients to any significant degree.

On the other hand, assume that $N \geq 3$ and that all parameters, except d are fixed. Hence, as d becomes arbitrarily large, we also see that the probability of finding a violation goes to zero. Putting into words:

for any $N \geq 3$, if the local dimension d is large enough, the vast majority of pure states will not violate any Bell inequality with bounded coefficients to any significant degree.

To sum up, we can formally state:

Corollary 2. *Let $|\psi\rangle \in \left(\mathbb{C}^d\right)^N$ be a unit vector distributed according to the uniform measure in the sphere S_{2d^N-1} of $\left(\mathbb{C}^d\right)^N$. Given integers $N \geq 2$ and $d \geq 2$, and given $b > 0$ the following statements below hold true:*

a) If d the local dimension satisfies $d > vm(2m-1)^2$, then

$$\mathbb{P}(V_{\mathrm{opt}} > c) \to 0, \text{ as } N \to \infty. \tag{7}$$

b) If N the number of parts satisfies $N \geq 3$, then:

$$\mathbb{P}(V_{\mathrm{opt}} > c) \to 0, \text{ as } d \to \infty. \tag{8}$$

So, in spite the fact that typically any N-partite pure state, with large N and/or d, is highly entangled, the typical value of the Bell violation is extraordinarily small.

II. THE PROOF

A. Idea of the proof

We first decompose the event $\{V_{\mathrm{opt}} > c\}$ into a union of pieces corresponding to each possible choice of POVM's and Bell inequality coefficients, as well as in violations from above and violations from below. This gives:

$$\mathbb{P}\left(|\psi\rangle : \sup_{\substack{A\in\mathscr{A}\\ T\in\mathscr{T}_b}} |Q(|\psi\rangle, T, A)| > c\right) = \mathbb{P}\left(\left[\bigcup_{\substack{A\in\mathscr{A}\\ T\in\mathscr{T}_b}} \{|\psi\rangle : Q(|\psi\rangle, T, A) > c\}\right] \bigcup \left[\bigcup_{\substack{A\in\mathscr{A}\\ T\in\mathscr{T}_b}} \{|\psi\rangle : Q(|\psi\rangle, T, A) < -c\}\right]\right) \tag{9}$$

$$= 2\mathbb{P}\left(\bigcup_{\substack{A\in\mathscr{A}\\ T\in\mathscr{T}_b}} \{|\psi\rangle : Q(|\psi\rangle, T, A) > c\}\right). \tag{10}$$

The first equality comes from the fact that the events considered are the same. In the disjoint in the *r.h.s* of Eq. (9), the fact that Q is linear in T guarantees that the two events have equal measure, which gives the second equality. Even though the event appearing on (10) is given by an infinite union of sets, the strategy is to replace it by a *finite* union without, however, changing too much its probability. This replacement is done trough suitable ε-nets for $\mathscr{A}$ and $\mathscr{T}$. We then just apply the union bound, together with a uniform bound on the probability distribution for the degree of violation for any *fixed* inequality with functional T and measurement settings A. This strategy results in the bound of Theorem 1 which has two terms as distinguished below:

$$\mathbb{P}(V_{\mathrm{opt}} > c) \leq 4 \underbrace{\left[\frac{8bNd^2(mv)^N}{\delta} + 2\right]^{mvNd^2}}_{(A)} \times \underbrace{\exp\left(-\frac{2d^N(c-\delta-1)^2}{36\pi^2(2m-1)^{2N-2}}\right)}_{(B)}, \tag{11}$$

The term (A) in Eq. (11) is just an estimate of the number of points of the $\varepsilon-$net. Term (B) is the uniform bound mentioned above. It results basically from Lévy's lemma [6] together with a pair of results on the smoothness of the Q function.

III. CONCLUSION

We have shown that for any fixed correlation scenario $\Gamma = (N,m,v)$ there exists an upper bound (see Eq. 3) for the typical violation that an $N-$partite pure state with local dimension d can exhibit. In particular, we have proved that, if the local dimension d is large enough relative to the complexity of the Bell scenario [7–9], then significant violations become extremely rare as N increases. More precisely, given a correlation scenario $\Gamma = (N,m,v)$, if

$$\frac{d}{m(2m-1)^2} > v, \tag{12}$$

then the probability of finding any significant violation of a Bell inequality whose coefficients are uniformly bounded [10] is extremely small for any $\delta > 0$. That is:

$$\mathbb{P}(V_{\text{opt}} \geq 1+\delta) \to 0, \text{ as } N \to \infty, \tag{13}$$

super-exponentially fast. This generalizes previous results by two of the present authors [11]. In addition, also surprisingly, we also have shown that if the number of parts is greater than two, then as the local dimension d goes to infinity, we also have the same behaviour for the probability of finding any significant violation, *i.e.*:

$$\mathbb{P}(V_{\text{opt}} \geq 1+\delta) \to 0, \text{ as } d \to \infty. \tag{14}$$

We note that the typicality arguments used here and in [11] are essential to several other results in quantum information theory [3, 7, 12–16])

Remarkably, our result stands in contrast with the fact that entanglement becomes typically large in the same limits of $N \to \infty$ or $d \to \infty$ we are taking. This further suggest that entanglement, though necessary, is not sufficient to explain non-local effects in quantum physics.

We now argue that the dependance of our result on the local dimension d is an essential feature of this kind of problem. In [12] the authors showed that, for correlation scenarios $\Gamma = (2,m,2)$, the typical behaviour of local correlations can be quite different depending on the value of d/m. Depending on how this ratio behaves, one may obtain (asymptotically in m) that correlation matrices do or do not display quantum effects. On the other hand, in our setting we are optimizing over all possible $POVM$'s. Therefore, it should be expected that in any contextuality scenario high violations of contextuality inequalities might be typical. If true, this is another significant difference between these two distinct situations [17–20].

Our approach is very general, in that it encompasses a large class of Bell inequalities. However, if arbitrary coefficients are allowed, it remains open whether a similar result would hold. Based on recent work [13] we believe that for general correlation scenarios $\Gamma = (N,m,v)$ it is possible to get rid of the uniformly boundedness requirement.

To conclude, we point out that our result also has implications in the context of the classical-to-quantum transition problem. As discussed firstly by Pitowsky (see [21]), there is an apparent contradiction between the fact that, on the one hand, multipartite pure quantum states rarely admit LHV models, since they always violate some Bell inequality [22]; and, on the other hand, the fact that in the macroscopic world such models are actually the rule. One way out of this is to invoke decoherence and claim that actual macroscopic systems should be described by highly mixed states, so that non-local correlations are not visible. Our result allows us to explain the same phenomenon from the (static) perspective of experimental feasibility. Indeed, even if a pure state is a good description for the macroscopic system, a typical state requires very intricate and sharp Bell test experiments in order to detect non-local correlations. This seems to be inconceivable in practice [23–25].

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