A diagrammatic axiomatisation of fermionic quantum circuits

Amar Hadzihasanovic
RIMS
Kyoto University
ahadziha@kurims.kyoto-u.ac.jp

Giovanni de Felice
Kang Feng Ng
Department of Computer Science
University of Oxford
giovanni.defelice@cs.ox.ac.uk
kangfeng.ng@cs.ox.ac.uk

This is an abstract of the paper *A diagrammatic axiomatisation of fermionic quantum circuits*, available at arXiv:1801.01231 [cs.LO]. The paper was accepted at the *Third International Conference on Formal Structures for Computation and Deduction* (FSCD 2018).

1 Introduction

The ZW calculus is a string-diagrammatic language for qubit quantum computing, introduced by the first author in [8]. Developing ideas of Coecke and Kissinger [4], it refined and extended the earlier ZX calculus [3, 1], while keeping some of its most convenient properties, such as the ability to handle diagrams as undirected labelled multigraphs. In the version of [9, Chapter 5], it provided the first complete equational axiomatisation of the monoidal category of qubits and linear maps, with the tensor product as monoidal product. Soon after its publication, the third author and Q. Wang derived from it a universal completion of the ZX calculus [11, 10].

Since its early versions, the ZX calculus has had the advantage of including familiar gates from the circuit model of quantum computing [12, Chapter 4], such as the Hadamard gate and the CNOT gate, as simple composite diagrams, which facilitates the transition between formalisms, and the application to known algorithms and protocols. This is related to the convenient access to complementary observables in the ZX calculus, in the guise of interacting special commutative Frobenius algebras [3].

This is not the case for the ZW calculus, that only includes one special commutative Frobenius algebra, corresponding to the computational basis, as a basic component. However, as noted already in [8], the ZW calculus has a fundamentally different "core", which is obtained by removing a single component that does not interact as naturally with the rest. This core has the property of only representing maps having a definite parity with respect to the computational basis, which happens to be compatible with an interpretation of the basis states of a single qubit as the empty and occupied states of a *local fermionic mode*, the unit of information of the *fermionic quantum computing* (FQC) model.

This connection suggested that an independent fermionic version of the ZW calculus could be developed, combining the best of both worlds with respect to FQC rather than qubit computing: the superior structural properties of the ZW calculus, including an intuitive normalisation procedure for diagrams, together with the superior hands-on features of the ZX calculus. In this paper, we define the category **LFM** capturing the FQC model as described in [2], we present a diagrammatic calculus inspired by the original ZW calculus and prove it is a complete axiomatisation for **LFM**.

Submitted to:
15th International Conference on Quantum Physics and Logic

2 The model

The mathematical underpinning of fermionic quantum processes is given by the following definitions.

Definition 1. A $\mathbb{Z}_2$-*graded Hilbert space* is a complex Hilbert space H decomposed as a direct sum $H_0 \oplus H_1$.

A *pure map* $f : H \to K$ of $\mathbb{Z}_2$-graded Hilbert spaces is a bounded linear map $f : H \to K$ such that $f(H_0) \subseteq K_0$ and $f(H_1) \subseteq K_1$ (*even* map), or $f(H_0) \subseteq K_1$ and $f(H_1) \subseteq K_0$ (*odd* map).

$\mathbf{Hilb}^{\mathbb{Z}_2}$ is the symmetric monoidal category of $\mathbb{Z}_2$-graded Hilbert spaces and pure maps with the tensor product as monoidal product.

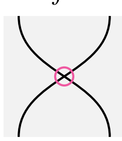
LFM is the full subcategory of $\mathbf{Hilb}^{\mathbb{Z}_2}$ with objects given by tensor powers of $B := \mathbb{C} \oplus \mathbb{C}$.

Fermionic quantum computing was introduced by Bravyi and Kitaev in [2]. In their presentation processes and states are distinguished: they give a universal gate set composed only of parity-preserving operations (even maps), but odd state preparations are allowed. This means that any pure map is allowed on fermionic modes, making **LFM** the appropriate model for fermionic circuits. In this interpretation the basic FQC resource is modelled by the $\mathbb{Z}_2$-graded Hilbert space B where $|0\rangle$ denotes an empty mode and $|1\rangle$ an occupied mode. These can be identified with the computational basis states of qubits, giving rise to an embedding of fermionic processes into qubit processes. This embedding was used to show that FQC is computationally equivalent to qubit computing [2], although the identification does not preserve locality and entanglement properties [7, 5].

3 The language

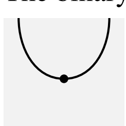
We begin by describing the generators of our language with their interpretation in **LFM**, starting with the *logical* components:

1. The *fermionic swap* gate:

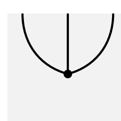


$$|00\rangle \mapsto |00\rangle, \qquad |10\rangle \mapsto |01\rangle,$$
$$|01\rangle \mapsto |10\rangle, \qquad |11\rangle \mapsto -|11\rangle.$$

2. The binary and ternary black vertex:

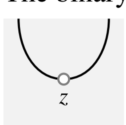


$$1 \mapsto |10\rangle + |01\rangle,$$



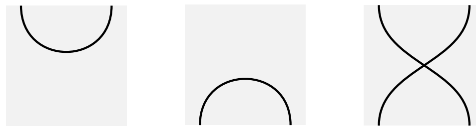
$$1 \mapsto |100\rangle + |010\rangle + |001\rangle.$$

3. The binary white vertex with parameter $z \in \mathbb{C}$:

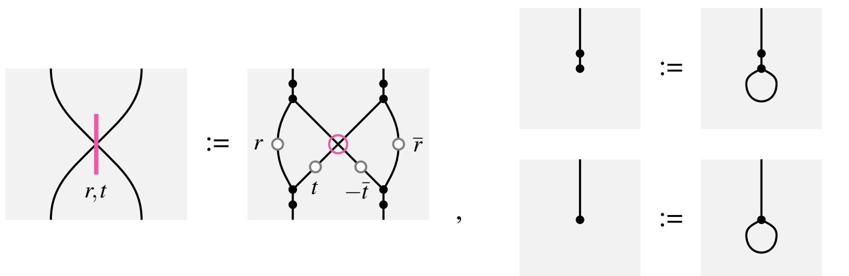


$$1 \mapsto |00\rangle + z|11\rangle.$$

The interpretations of binary and ternary black vertices are known as EPR state and W state, respectively, in qubit theory [6]. White vertices with parameter $z = e^{i\alpha}$ correspond to phases in the computational basis. In addition to the logical components above, we need the following *structural* components — the *dualities* and the *swap* — which allow us to treat all our diagrams as components of a circuit diagram, which can be connected together in an undirected fashion, permuting and transposing their inputs and outputs:



The main physical gates of interest in FQC are then either basic components (such as the fermionic swap and phase gates) or simple composite diagrams such as the beam splitter with parameters r, t, or the empty and occupied state preparations:



We provide axioms for the fermionic ZW calculus, characterising the way the logical components interact with each other. The black vertices are the operational core of the language, characterised by the *fermionic line*, an algebra that forms a Hopf algebra with its own transpose. As in the original ZW calculus, the convolution algebra of the fermionic line superposes (adds) phases and composition multiplies them. The main novelty here lies in the axiomatisation of the fermionic swap, which ties together all the other components.

Finally we prove our main theorem, that the fermionic ZW calculus is an axiomatisation of **LFM**. We achieve this by defining a normal form for diagrams, from which one can easily read the interpretation in **LFM**, and showing that any diagram can be rewritten in normal form using the axioms.

4 Conclusion

We have constructed a complete axiomatisation of fermionic quantum circuits, by extracting a core fragment of the ZW calculus. The main advantage of this approach is that it allows to reason rigorously about FQC circuits, and prove correctness of protocols with algebraic manipulations of diagrams. Replacing matrix multiplication with diagrammatic moves simplifies the reasoning and makes proofs more intuitive, as the diagrams resemble operationally the physical processes they encode. As an application, we explored the connection of FQC to linear optics, using the decomposition of the beam splitter given above to derive the statistics of the fermionic Mach-Zehnder interferometer.

References

[1] M. Backens (2014): *The ZX-calculus is complete for stabilizer quantum mechanics*. New Journal of Physics 16(9), p. 093021, doi:10.1088/1367-2630/16/9/093021. Available at `https://doi.org/10.1088%2F1367-2630%2F16%2F9%2F093021`.

[2] S.B. Bravyi & A. Yu. Kitaev (2002): *Fermionic Quantum Computation*. *Annals of Physics* 298(1), pp. 210–226, doi:10.1006/aphy.2002.6254.

[3] B. Coecke & R. Duncan (2008): *Interacting quantum observables*. In: *Automata, Languages and Programming*, Springer, pp. 298–310, doi:10.1007/978-3-540-70583-3_25.

[4] B. Coecke & A. Kissinger (2010): *The Compositional Structure of Multipartite Quantum Entanglement*. In: *Automata, Languages and Programming*, Springer Berlin Heidelberg, pp. 297–308, doi:10.1007/978-3-642-14162-1˙25. Available at https://doi.org/10.1007%2F978-3-642-14162-1_25.

[5] G.M. D'Ariano, F. Manessi, P. Perinotti & A. Tosini (2014): *The Feynman problem and fermionic entanglement: Fermionic theory versus qubit theory*. *International Journal of Modern Physics A* 29(17), p. 1430025, doi:10.1142/s0217751x14300257.

[6] W. Dür, G. Vidal & J.I. Cirac (2000): *Three qubits can be entangled in two inequivalent ways*. *Physical Review A* 62(6), doi:10.1103/physreva.62.062314.

[7] N. Friis, A.R. Lee & D.E. Bruschi (2013): *Fermionic-mode entanglement in quantum information*. *Physical Review A* 87(2), doi:10.1103/physreva.87.022338.

[8] A. Hadzihasanovic (2015): *A Diagrammatic Axiomatisation for Qubit Entanglement*. In: *Proceedings of the 30th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '15, IEEE, pp. 573–584, doi:10.1109/LICS.2015.59.

[9] A. Hadzihasanovic (2017): *The algebra of entanglement and the geometry of composition*. Ph.D. thesis, University of Oxford.

[10] A. Hadzihasanovic, K.F̃. Ng & Q. Wang (2018): *Two complete axiomatisations of pure-state qubit quantum computing*. Accepted at the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS) 2018.

[11] K.F. Ng & Q. Wang (2017): *A universal completion of the ZX-calculus*. *arXiv preprint arXiv:1706.09877*.

[12] M.A. Nielsen & I.L. Chuang (2009): *Quantum Computation and Quantum Information*. Cambridge University Press, doi:10.1017/cbo9780511976667. Available at https://doi.org/10.1017%2Fcbo9780511976667.