A Categorical Reconstruction of Quantum Theory (Extended Abstract)

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The Hilbert space formulation of quantum theory has remained difficult to interpret ever since it was first described by von Neumann [19]. Over the years this has led to numerous attempts to understand the quantum world from more basic, operational or formal principles. Most recently there has been a great interest in 'reconstructions' of quantum theory as a theory of information [11, 4, 12, 21, 20, 14]. In these, quantum theory is singled out via operational axioms, referring to the likelihoods assigned to experimental procedures, from amongst all such general probabilistic theories.

Typically, a central aspect of any such theory is the specification of certain allowed physical *systems* and *processes* between them, which may be composed like pieces of apparatus in a laboratory. It is well-known that such 'process theories' correspond precisely to *monoidal categories* [9], very general mathematical structures which come with an intuitive graphical calculus allowing one to reason using 'circuit diagrams' [16]. In the usual approach to reconstructions, one then explicitly adds further probabilistic structure using an assumption known as *finite tomography*, which enforces that the processes of any given type generate a finite-dimensional real vector space.

However, there is a second tradition in the foundations literature characterised by avoiding these tomography assumptions, instead studying physical theories such as quantum theory purely in terms of their diagrammatic or categorical aspects, and collectively referred to as *categorical quantum mechanics* (CQM) [2]. The categorical approach has proven highly successful in studying numerous aspects of quantum foundations and computing [10, 7, 1, 6]. As such it is natural to ask whether one may recover quantum theory itself in this framework, and a reconstruction theorem for CQM has long been desired [8]. In the full version[1] of this article, we present such a category-theoretic reconstruction of quantum theory.

The Framework We work in the framework of *dagger theories*. Firstly, such a theory specifies a symmetric monoidal category $(\mathbf{C}, \otimes)$, in which morphisms may be represented by boxes with input and output wires labelled by objects. In particular there is a trivial system I corresponding to the empty diagram, and we call morphisms of the form $I \to A$, $A \to I$ and $I \to I$ *states, effects* and *scalars*, respectively. Moreover we require that each object comes with a distinguished effect denoted ⏚_A which we think of as simply *discarding* a system, and suppose the presence of a *dagger* $\dagger$ operation allowing us to 'reverse' morphisms and so flip diagrams upside-down.

Along with this the theories we consider are *compact* allowing us to bend wires in our diagrams and thus exchange inputs and outputs of morphisms, and come with *zero morphisms* denoted by 0. It will be useful to mention some definitions; we call an object A *non-trivial* when $\mathrm{id}_A \neq 0$ and ⏚_A is not an isomorphism, a pair of states $|0\rangle, |1\rangle$ of the same object *orthonormal* when $|0\rangle^\dagger \circ |0\rangle = \mathrm{id}_I = |1\rangle^\dagger \circ |1\rangle$ and $|1\rangle^\dagger \circ |0\rangle = 0$, and a morphism f *causal* when it satisfies $\text{⏚} \circ f = \text{⏚}$.

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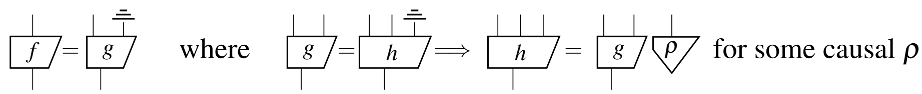
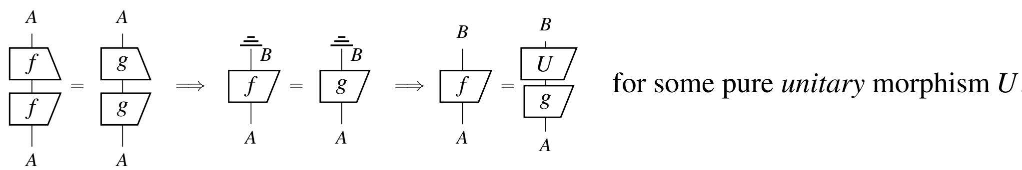
The Principles

We say that a compact dagger theory **C** satisfies the *operational principles* when it contains a non-trivial object and satisfies the following.

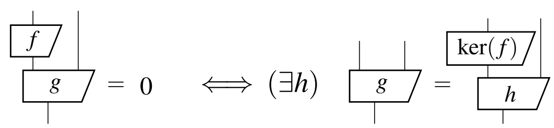
1. *Purification.* Any non-zero morphism f has a dilation g which is *pure* in the sense that it has only trivial dilations:



   Further we require that every non-zero object has a causal pure state which is *causal*, that pure morphisms form a dagger monoidal subcategory containing all zero morphisms, and that purifications are *essentially unique* [3] and form an *environment structure* [5] amounting to:



2. *Kernels.* Every morphism f has a *dagger kernel*, that is, an *isometry* $\ker(f)$ satisfying



(1)

   Moreover for any such kernel k we define $k^\perp := \ker(k^\dagger)$ and require that any morphisms f, g having equal composites with each of $\bar{\top} \circ k^\dagger$ and $\bar{\top} \circ k^{\perp\dagger}$ in fact have $\bar{\top} \circ f = \bar{\top} \circ g$.
3. *Pure Exclusion.* Every causal pure state ψ of a non-trivial object has $f \circ \psi = 0$ for some non-zero morphism f.
4. *Conditioning.* For every pair of orthonormal states $|0\rangle, |1\rangle$ and every pair of states ρ, σ of some object there is a morphism f with $f \circ |0\rangle = \rho$ and $f \circ |1\rangle = \sigma$.

This last principle is very mild, and when the the others hold is in fact equivalent to the presence of a basic operational feature, namely the ability to 'coarse-grain' processes using a well-behaved addition operation $f + g$. In particular, this makes the collection of scalars $R = \mathbf{C}(I, I)$ a semi-ring. Finally, we will call R *bounded* when no scalar r has that for all $n \in \mathbb{N}$ there is some r_n with $r = n + r_n$; for example the positive reals $\mathbb{R}^+$ are certainly bounded.

Operational motivation for the principles comes from their resemblance to those of the reconstruction due to Chiribella, D'Ariano and Perinotti (CDP) [4], being essentially a reformulation of these for the setting of dagger compact categories, under the following correspondence:

CDP Axioms	**Categorical Features**
Coarse-graining	Conditioning
Causality	Discarding $\bar{\top}$
Atomicity of composition	Environment structure
Purification	Environment structure
Perfect distinguishability	Kernels + pure exclusion
Ideal compressions	Kernels + pure exclusion
Essential uniqueness	Essential uniqueness

Since our framework extends beyond probabilistic theories, our results will apply to not only quantum theory, but a broader class of generalised quantum-like theories.

Examples. *The dagger theory* **Quant** *has as objects finite-dimensional Hilbert spaces and as morphisms completely positive linear maps* $f\colon B(\mathscr{H}) \to B(\mathscr{K})$*, with discarding given by the trace. More generally, for any commutative involutive semi-ring* $(S,\dagger)$ *let* $\mathbf{Mat}_S$ *be the category of S-valued matrices. Then using Selinger's* CPM construction *[15], we define a theory* $\mathbf{Quant}_S := \mathsf{CPM}(\mathbf{Mat}_S)$. *Whenever S is a field with suitable properties,* $\mathbf{Quant}_S$ *satisfies our principles. In particular so do* $\mathbf{Quant} \simeq \mathbf{Quant}_{\mathbb{C}}$ *and the quantum theory* $\mathbf{Quant}_{\mathbb{R}}$ *on* real Hilbert spaces *[17, 13].*

The Reconstruction

Our main result is then the following.

Theorem. *Let* **C** *be a dagger theory satisfying the operational principles, with scalars R. Then there is an embedding of theories* $\mathbf{Quant}_S \hookrightarrow \mathbf{C}$ *for some commutative involutive ring S with* $R \simeq \{s^\dagger \cdot s \mid s \in S\}$. *Moreover if R is bounded this is an equivalence of theories* $\mathbf{C} \simeq \mathbf{Quant}_S$.

The relationship between R and S generalises that between $\mathbb{R}^+$ and $\mathbb{C}$ in the case of **Quant**, and we can greatly strengthen this analogy under one extra assumption. Firstly, it in fact follows that R has cancellative addition and so may be freely extended to a ring, denoted $D(R)$. Let us then write $D(R)[i]$ for the involutive ring with elements $a + b \cdot i$ for $a, b \in D(R)$, where $1 = -i^2 = i \cdot i^\dagger$, and $a^\dagger = a$ for $a \in D(R)$.

Proposition. *If* $R = \mathbf{C}(I,I)$ *is bounded and has square roots of all elements then the involutive ring S is equivalent to either* $D(R)$ *or* $D(R)[i]$.

Finally, specialising to the typical case where **C** is *probabilistic*, meaning that it comes with an isomorphism of semi-rings $\mathbf{C}(I,I) \simeq \mathbb{R}^+$, we immediately obtain the following.

Corollary. *Any probabilistic theory satisfying the above principles is equivalent to* $\mathbf{Quant}_{\mathbb{R}}$ *or* $\mathbf{Quant}_{\mathbb{C}}$.

These results are unlike previous reconstructions in that they are entirely category-theoretic, not assuming any form of tomography or even any linear structure, and treat probabilistic theories only as a special case. In particular avoiding the use of *local tomography* [13] has allowed us to recover both standard quantum theory and that over real Hilbert spaces.

The Proof Our technique is based on a novel approach to describing superpositions in general process theories due to the author in [18]. Previously, such as in [2], these have been modelled using *biproducts* in the category **Hilb** of Hilbert spaces and linear maps. However, only the morphisms in its quotient $\mathbf{Hilb}_\sim$ after identifying global phases have a direct physical interpretation. To describe superpositions in the latter category we introduce the new notion of a *phased biproduct*. In fact in [18] it is shown that from any suitable category **B** possessing these one may construct a new category $\mathsf{GP}(\mathbf{B})$ with biproducts; for example $\mathsf{GP}(\mathbf{Hilb}_\sim) \simeq \mathbf{Hilb}$. This provides a general 'recipe' for quantum reconstructions which lies at the heart of our proof and we hope to be applicable to further reconstruction results in the future.

Future Work Our results are suggestive of a new approach to reconstructing physical theories purely in terms of their process-theoretic properties, and there are many avenues for future research. Most notably, it would now be desirable to avoid, or rather derive, the dagger operation, which lacks a clear interpretation for general mixed processes. This would provide a reconstruction which is fully operational, as well as categorical, and allow infinite-dimensional systems to be considered. Connections with other quantum reconstructions also remain to be explored; notable are the recent probabilistic reconstructions due to Coecke, Selby and Scandolo [14] and van de Wetering [20].

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