Diagrammatic Reasoning beyond Clifford+T Quantum Mechanics⋆

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Abstract. The ZX-Calculus is a graphical language for diagrammatic reasoning in quantum mechanics and quantum information theory. An axiomatisation has recently been proven to be complete for an approximatively universal fragment of quantum mechanics, the so-called Clifford+T fragment. We focus here on the expressive power of this axiomatisation beyond Clifford+T Quantum mechanics. We consider the full pure qubit quantum mechanics, and mainly prove two results: (i) First, the axiomatisation for Clifford+T quantum mechanics is also complete for all equations involving some kind of *linear* diagrams. The linearity of the diagrams reflects the phase group structure, an essential feature of the ZX-calculus. In particular all the axioms of the ZX-calculus are involving linear diagrams. (ii) We also show that the axiomatisation for Clifford+T is not complete in general but can be completed by adding a single (non linear) axiom, providing a simpler axiomatisation of the ZX-calculus for pure quantum mechanics than the one recently introduced by Ng&Wang.

The ZX-calculus, introduced by Coecke and Duncan [5] is a graphical language for pure state qubit quantum mechanics. The ZX-calculus has multiple applications in quantum information theory [6], including the foundations [2,8], measurement-based quantum computation [7,11,15] or quantum error correcting codes [3,4,9,10], and can be used through the interactive theorem prover Quantomatic [17,18].

The ZX-calculus is universal: any quantum evolution can be represented by a ZX-diagram. ZX-diagrams are parametrised by angles, and various fragments of the language have been considered, based on some restrictions on the angles: the $\frac{\pi}{p}$-fragment consists in considering only the diagrams made with angles multiple of $\frac{\pi}{p}$. The $\frac{\pi}{2}$-fragment (resp. π-) corresponds to stabilizer quantum mechanics (resp. real stabilizer quantum mechanics) and are not universal for quantum mechanics, even approximately. The $\frac{\pi}{4}$-fragment corresponds to the so called Clifford+T quantum mechanics and is approximately universal: any quantum evolution can be approximated in this fragment with arbitrary accuracy.

The ZX-calculus also comes with a powerful axiomatisation which can be used to transform a diagram into another diagram representing the same quantum evolution. The axioms of the ZX-calculus are given in Figure 1. Some of the axioms are parametrised by variables, meaning that the axioms are true for all possible values of these variables. Notice that all the variables are used in a linear fashion, i.e. all the angles are some linear combinations of variables and constants, like in (S1) or (SUP) for instance. The use of such linear diagrams in the axiomatisation captures the phase group structure, one of the two fundamental quantum features (with the complementary observables) of the ZX-calculus [5].

Completeness of the axiomatisation is an essential feature: the axiomatisation is complete if for any pair of diagrams representing the same quantum evolution, one

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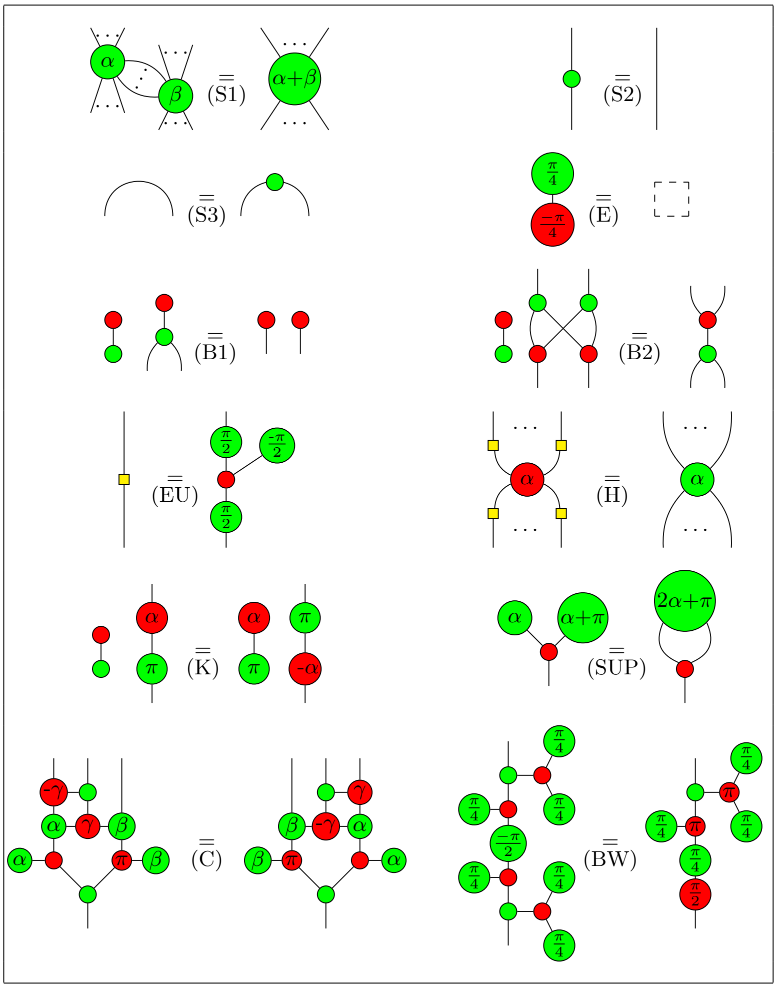


Fig. 1. Set of rules for the Clifford+T ZX-Calculus with scalars. All of these rules also hold when flipped upside-down, or with the colours red and green swapped. The right-hand side of (E) is an empty diagram. (...) denote zero or more wires, while (.⋰) denote one or more wires.

can use the axioms of the language to transform one diagram into the other. The ZX-calculus has been proved to be complete for the π- and $\frac{\pi}{2}$-fragments of the ZX-calculus [1,12]. Recently the axiomatisation given in Figure 1 has been proved to be completed for the $\frac{\pi}{4}$-fragment, providing the first complete axiomatisation for an approximately universal fragment of the ZX-calculus [16]. This last result relies on the completeness of another graphical language which represents integer matrices, called ZW-Calculus [13]. The ZW-Calculus has since been extended to represent all matrices over $\mathbb{C}$ [14]. This achievement gave hope for a universal completion of the ZX-Calculus, and soon enough, a first result appeared [19]. To make the ZX-calculus complete for the full quantum mechanics, two new generators and a large amount of axioms (32 axioms versus 12 for the axiomatisation for Clifford+T quantum mechanics) have been introduced, some of them being non linear.

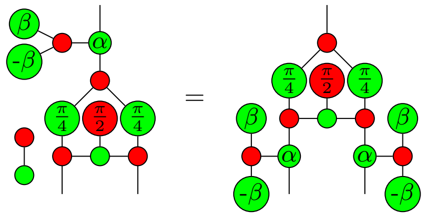
One can wonder whether this result can be improved. We address this question in two steps: (i) First, we prove that the complete axiomatisation for Clifford+T quantum mechanics can also be used to prove a significant amount of equations beyond this fragment: all true equations involving diagrams which are linear with constants multiple of $\frac{\pi}{4}$ can be derived:

Theorem 1. *For any ZX-diagrams* $D_1(\boldsymbol{\alpha})$ *and* $D_2(\boldsymbol{\alpha})$ *linear in* $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_k$ *with constants in* $\frac{\pi}{4}\mathbb{Z}$*,*

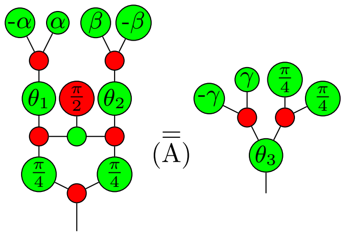
$$\forall \boldsymbol{\alpha} \in \mathbb{R}^k, [\![D_1(\boldsymbol{\alpha})]\!] = [\![D_2(\boldsymbol{\alpha})]\!] \;\Leftrightarrow\; \forall \boldsymbol{\alpha} \in \mathbb{R}^k, \mathrm{ZX} \vdash D_1(\boldsymbol{\alpha}) = D_2(\boldsymbol{\alpha})$$

We even refine this result, by taking into account the *multiplicity* μ of the variables in an equation (formally defined in the paper), which basically counts the number of useful occurrences of the variables. We can use the previous theorem if we have proven $[\![D_1(\boldsymbol{\alpha})]\!] = [\![D_2(\boldsymbol{\alpha})]\!]$ for a sufficient number of values of $\boldsymbol{\alpha}$, which correlates to the multiplicity. This implies that some general derivation can be induced from a finite-case-based reasoning. We point out with several examples that this result can be used to derive some new non-trivial equations. E.g.:

$\forall \alpha, \beta \in \mathbb{R}, \quad \mathrm{ZX} \vdash$



(ii) Then we show that this axiomatisation is not complete in general, and we propose a complete axiomatisation for the full pure qubit quantum mechanics which consists in adding a single (non-linear) axiom:



$$2e^{i\theta_3}\cos(\gamma) = e^{i\theta_1}\cos(\alpha) + e^{i\theta_2}\cos(\beta)$$

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