Contextual advantage for state discrimination

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We identify quantitative limits on the success probability for minimum error state discrimination in any noncontextual ontological model. These limits constitute experimentally testable noncontextuality inequalities. This work proves that contextuality is a resource for quantum state discrimination, and provides a first step towards showing that contextuality is the resource for other quantum information-processing tasks such as bit commitment and coin flipping.

This work summarizes Ref. [1], namely Phys. Rev. X **8**, 011015.

1 Introduction

Understanding the boundary between the quantum and the classical is of fundamental importance for understanding quantum theory. One successful metric for nonclassicality was proposed by Kochen-Specker [2] and Bell [3], and has since been significantly refined and generalized [4]. It is the generalized notion of noncontextuality from Ref. [4] which we study in this paper. Generalized noncontextuality subsumes many other pre-existing notions of nonclassicality, such as the negativity of quasi-probability representations [5], the generation of anomalous weak values [6], and violations of local causality [4]. The quantum-classical boundary is also of practical importance in identifying tasks which admit of a quantum advantage. For example, the failure of noncontextuality has been shown to be a resource, leading to advantages for cryptography [7, 8, 9] and computation [10, 11, 12].

In Ref. [1], we analyze minimum-error state discrimination (MESD) through the lens of noncontextuality. Quantum state discrimination is a task wherein one must guess which quantum state describes a given quantum system when the state of that system is drawn from a known set of possibilities with a known prior distribution, and the estimation is based on the outcome of a measurement of one's choosing. In "minimum error" state discrimination, the objective is to minimize the probability that the estimate is in error. We focus on the simplest case of a set containing two states having equal a priori probability.

Aspects of quantum state discrimination are often asserted to be intrinsically nonclassical phenomena, but these claims have not previously been justified by a rigorous no-go theorem. Taking generalized noncontextuality as one's principle of classicality, we prove such a no-go theorem. In doing so, we find a strongly nonclassical aspect of minimum error state discrimination: the particular dependence of the probability of successful discrimination on the overlap of the quantum states. For a given overlap, the quantum probability of discrimination is larger than can be accounted for by a noncontextual model. Further, we derive noise-robust noncontextuality inequalities for this no-go theorem, which can experimentally witness this contextual advantage for state discrimination, independently of the validity of quantum theory and even in the presence of noise.

Because quantum state discrimination is a primitive in many important quantum information processing protocols [13, 14], this work constitutes a first step towards identifying contextuality as a resource for more tasks concerning communication, computation, and cryptography. For example, protocols for

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bit commitment and coin flipping rely sensitively on one's ability to discriminate quantum states. Our work thus constitutes a first step towards proving that these tasks are powered by contextuality.

2 Minimum Error State Discrimination in Quantum Theory

The basic operational scenario we are interested in characterizing—that of minimum error state discrimination—is shown in Fig. 1 (a). Given a single copy of a quantum state chosen at random from the set $\{ket\psi, |\phi\rangle\}$, one is tasked with guessing the identity of the state, by performing any measurement on the state. The measurement scheme that yields the greatest probability of guessing correctly which of two nonorthogonal states was prepared is called the *minimum error* state discrimination (MESD) scheme.

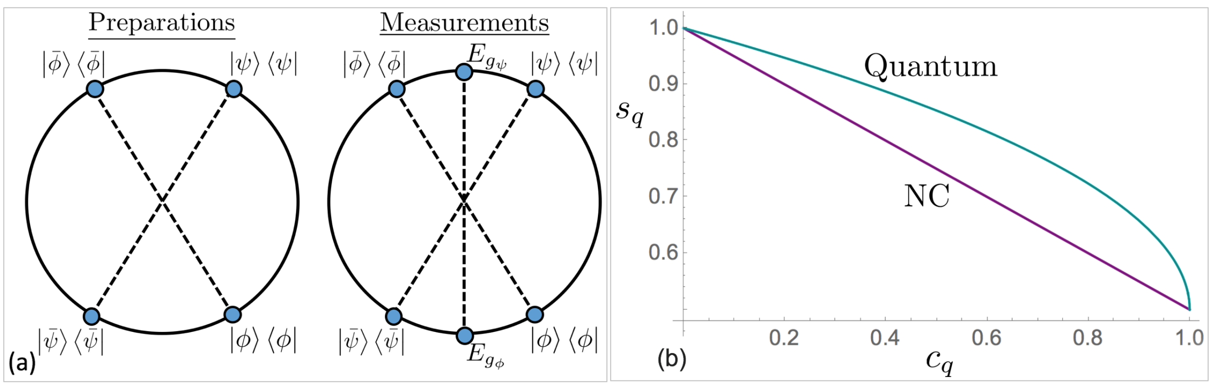

Figure 1: (a) The quantum states and measurements in our scenario, depicted as vectors in a plane of the Bloch ball. (b) The optimal success rate at MESD in quantum theory and in any noncontextual model.

Since $|\phi\rangle$ and $|\psi\rangle$ are prepared with equal probability, the POVM $\{E_{g_\phi}, E_{g_\psi}\}$ achieving MESD is the one consisting of projectors onto the basis that straddles $|\phi\rangle$ and $|\psi\rangle$ in Hilbert space, which is depicted in the Bloch sphere in Fig. 1. This is called the *Helstrom measurement* [15]. It is well-known that the probability of guessing the state correctly using the Helstrom measurement is

$$s_q = \frac{1}{2}(1+\sqrt{1-|\langle\phi|\psi\rangle|^2}) = \frac{1}{2}(1+\sqrt{1-c_q}), \tag{1}$$

where c_q is an operational measure of the 'overlap' between state ψ and ϕ. Specifically, c_q quantifies the probability of a state ψ passing the test for a state ψ:

$$c_q = \mathrm{Tr}[|\phi\rangle\langle\phi|\psi\rangle\langle\psi|] = |\langle\phi|\psi\rangle|^2. \tag{2}$$

	$\vert\phi\rangle\langle\phi\vert$	$\vert\psi\rangle\langle\psi\vert$	$\vert\bar{\phi}\rangle\langle\bar{\phi}\vert$	$\vert\bar{\psi}\rangle\langle\bar{\psi}\vert$
$\vert\phi\rangle\langle\phi\vert$	1	c_q	0	$1-c_q$
$\vert\psi\rangle\langle\psi\vert$	c_q	1	$1-c_q$	0
E_{g_ϕ}	s_q	$1-s_q$	$1-s_q$	s_q

Table 1: Data table in the ideal quantum case.

We also consider the states $|\bar{\phi}\rangle$ and $|\bar{\psi}\rangle$, which are orthogonal to $|\phi\rangle$ and $|\psi\rangle$. By symmetry, the Helstrom measurement for $|\phi\rangle$ and $|\psi\rangle$ is also the optimal measurement for discriminating $|\bar{\phi}\rangle$ vs. $|\bar{\psi}\rangle$. As such, this additional structure does not require specifying any additional facts about the phenomenology of quantum state discrimination. However, the inclusion of $|\bar{\phi}\rangle$ and $|\bar{\psi}\rangle$ in our analysis provides us with a nontrivial operational equivalence relation among the preparations, namely,

$$\frac{1}{2}|\phi\rangle\langle\phi| + \frac{1}{2}|\bar{\phi}\rangle\langle\bar{\phi}| = \frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}|\bar{\psi}\rangle\langle\bar{\psi}| = \frac{1}{2}\mathbb{I}. \tag{3}$$

This equivalence relation, together with the phenomenology of quantum state discrimination (summarized in Table 1) is sufficient to derive a no-go theorem for noncontextuality.

3 Minimum Error State Discrimination in a Noncontextual Model

An ontological model for an experiment specifies a set of states of reality, and a preparation (e.g. of a quantum state) is presumed to sample from a fixed distribution over these states. Each outcome of a measurement is presumed to occur according to some fixed probability which is a function only of the state of reality, which mediate correlations between the preparations and the measurement outcomes.

Our no-go theorem relies on preparation noncontextuality [4], the assumption that laboratory preparations which have the same statistics for all measurements must have identical representations in the ontological model of one's experiment. For example, Eq. (3) describes two such preparation procedures in quantum theory: the equal mixture of $|\phi\rangle$ and $|\psi\rangle$ cannot be distinguished from the equal mixture of $|\bar{\phi}\rangle$ and $|\bar{\psi}\rangle$ by *any* measurement. Noncontextuality demands identical ontological representations for these two ensemble-average preparations, providing the constraint that leads to our main theorem.

In Ref. [1], we show that any ontological model which reproduces the form of the data in Table 1 and is preparation noncontextual with respect to Eq. (3) *must* satisfy a trade-off between s_q and c_q:

$$s_q \leq 1 - \frac{c_q}{2}; \tag{4}$$

that is, the probability of success in minimum error state discrimination is bounded by one minus half the confusability of the quantum states. This tradeoff relation is plotted in Fig. 1 (b). Clearly, quantum theory allows for better discrimination of $|\phi\rangle$ and $|\psi\rangle$ than is possible in any noncontextual model.

3.1 Generalizations and Additional Results

Eq. (4) is not a directly testable noncontextuality inequality, since its proof [1] relies on the quantum formalism and also on the idealization that certain probabilities in Table 1 are exactly 0 or 1. However, in Ref. [1], we drop both of these idealizations and derive an entirely operational result which bounds the probability of state discrimination in any noncontextual model as a function of the confusability *and* a noise parameter which quantifies the deviation from these idealized values of 0 and 1. This bound constitutes a noncontextuality inequality which witnesses a contextual advantage for state discrimination.

Our work makes several other contributions to the understanding of state discrimination and of generalized noncontextuality. First, we prove an isomorphism between our operational MESD scenario and a two-party Bell test. We also generalize existing methods [16] for simulating exact operational equivalences in a real experiment. Most importantly, we introduce (for the first time) a novel algorithm for deriving the full set of necessary and sufficient noncontextuality inequalities for *any* finite prepare-and-measure scenario, with respect to any fixed operational equivalences.

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