Boolean subalgebras of orthoalgebras (extended abstract)

John Harding Chris Heunen Bert Lindenhovius Mirko Navara

This is a summary of the manuscript available at arXiv:1711.03748v2.

Introduction *Contextuality* is the phenomenon in quantum theory that the outcome of a measurement may depend on the context in which that measurement is made, *i.e.* on the experimental implementation of that measurement. This principle prevents hidden-variable explanations and clarifies why deterministic explanations of quantum theory are impossible. In general, contextuality concerns how various parts, that are locally consistent, fit together globally. In this work, we show how "the shape of how the parts fit together" is enough to determine the whole. The (contents of the) parts themselves are not necessary.

Contextuality can be made rigorous in various ways, including: in terms of Bell inequalities; as graphs of correlations [2]; or as a sheaf condition [1]. Here, we follow an algebraic approach, where the quantum system is modeled as an algebra, and measurement contexts are modeled as certain structured subalgebras. Perhaps the most well-known work in this direction [9] uses C*-algebras for quantum systems, and commutative C*-subalgebras for measurement contexts. For simplicity we disregard analytic details and consider various kinds of discrete algebras, such as *orthoalgebras*, effect algebras, and orthomodular posets. The appropriate notion of a measurement context then is a *Boolean* subalgebra.

Boolean subalgebras Our main object of study is the partially ordered set $\mathrm{BSub}(A)$ of Boolean subalgebras $B \subseteq A$ of an orthoalgebra A. This fits in the established mathematical pattern where some collection of substructures of a structure plays a key role: in classical logic, the collection of subsets of a set; in probability theory, the measurable subsets of a measurable space; in intuitionistic logic, the open subsets of a topological space; in projective geometry, the subspaces of a vector space; and in quantum theory, the collection of closed subspaces of a Hilbert space.

Let us emphasise again that, if an orthoalgebra A is the 'whole', we merely consider the 'shape' $\mathrm{BSub}(A)$ of how the 'parts' $B \subseteq A$ fit together, and not the internal structure of the 'parts' B as Boolean algebras. To solve a jigsaw puzzle, you just need to find out how the pieces fit, and don't need the picture on the pieces. For example, if A is a Boolean algebra with 4, 8, or 16 elements, $\mathrm{BSub}(A)$ looks as follows.

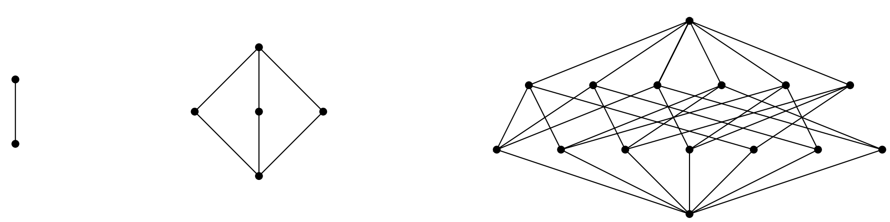


Figure 1: The posets $\mathrm{BSub}(A)$ for Boolean algebras A with 4, 8, and 16 elements.

A Boolean algebra with 4 elements has two subalgebras: $\{0,1\}$, and itself. A Boolean algebra A with 8 elements has five subalgebras: $\{0,1\}$, three subalgebras $\{0,a,a',1\}$ for $a \in A \setminus \{0,1\}$, and A itself. A Boolean algebra with 16 elements has a more complicated structure of subalgebras, containing subalgebras with 2, 4, 8, and 16 elements.

Submitted to:
QPL 2018

Related work This work should be considered as a middle stage of a larger project. There has been considerable work showing that $\mathrm{BSub}(A)$ determines A in the setting of Boolean algebras [12], orthomodular posets [7], von Neumann algebras [3], and with various types of C*-algebras [4, 6, 10]. In the analytic cases, it is the Jordan structure that is determined. However, these results are all of the following nature: if $\mathrm{Sub}(A)$ and $\mathrm{Sub}(A')$ are isomorphic, then there exists a (Jordan) isomorphism between A and A'. Even when A is a Boolean algebra, the only known method to reconstruct A from $\mathrm{BSub}(A)$ is indirect, via a family of colimits in the category of Boolean algebras [5, 8].

Construction We reconstruct A from $\mathrm{BSub}(A)$ directly, using a new notion of *direction* for an atom of $\mathrm{BSub}(A)$. We show that if A is a Boolean algebra, orthomodular poset, or orthoalgebra, then the directions of $\mathrm{BSub}(A)$ can be organized into an algebra isomorphic to A. This process somewhat resembles the reconstruction of a sober topological space from the points of its lattice of open sets.

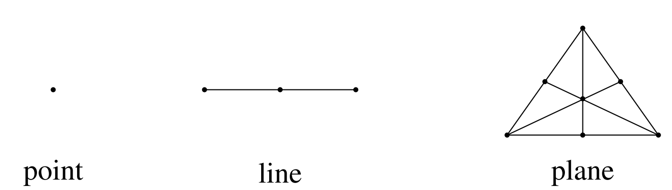
Categorical characterisation Moreover, we characterise the posets that arise as $\mathrm{BSub}(A)$ for some A. For the classical case of Boolean algebras A such a characterisation is known [5]: the posets $\mathrm{BSub}(A)$ are those algebraic lattices where the principal downset of each compact element is a partition lattice. Such lattices are called Boolean domains. We extend this to the quantum setting of orthoalgebras.

For orthoalgebras A we identify several basic properties of $\mathrm{BSub}(A)$, such as having Boolean domains as principal downsets. We call such posets *orthodomains*. It would take complex combinatorics to characterise in elementary terms the orthodomains of the form $\mathrm{BSub}(A)$ from some orthoalgebra A. We sidestep this issue by characterising them as the orthodomains with enough directions, just like topological spaces are frames with enough points. Given an orthodomain P with enough directions, we make its directions into an orthoalgebra $\mathrm{Dir}(P)$, and show that up to some small pathologies:

$$\mathrm{Dir}(\mathrm{BSub}(A)) \simeq A \qquad\qquad \mathrm{BSub}(\mathrm{Dir}(P)) \simeq P$$

Thus there is a correspondence between orthoalgebras and orthodomains with enough directions. We extend this *functorially* by defining morphisms between orthodomains as certain partial maps between their atoms. The functors BSub and Dir form an equivalence, apart only from technicalities to avoid obstacles of small algebras: 1- and 2-element Boolean algebras have the same lattice of subalgebras; and a 4-element Boolean algebra has 2 automorphisms while its lattice of subalgebras has only one. This categorical viewpoint is new, even in the setting of Boolean algebras and Boolean domains.

Representation Thus we may work with $\mathrm{BSub}(A)$ directly, instead of with A itself, without losing information. This may be more tractable than at first expected. We show that for orthoalgebras A, the poset $\mathrm{BSub}(A)$ is determined by its elements of height 3 or less, and for any Boolean algebra or orthomodular poset A, the poset $\mathrm{BSub}(A)$ is even determined by its elements of height 2 or less. In the latter setting it is fruitful to think of projective geometry, treating atoms of $\mathrm{BSub}(A)$ as *points*, and elements of height 2 in $\mathrm{BSub}(A)$ as *lines*. Each line is an 8-element subalgebra, hence will contain exactly three 4-element subalgebras, hence three points. The hypergraphs for the example in Figure 1 then look as follows:



The standard representation of orthomodular posets and orthoalgebras is via *Greechie diagrams* [11]. Both methods are preferable to Hasse diagrams, which have no convenient way to indicate orthocomplements or orthogonal joins, and are completely intractable in all but the simplest cases. Greechie diagrams will generally be much smaller than our hypergraphs, but the price to pay is that they hide much of the structure, often in a way that is very difficult to understand. Furthermore, Greechie diagrams apply only to chain-finite orthomodular posets and orthoalgebras, while hypergraphs apply to arbitrary ones, even ones without atoms. Finally, hypergraphs allow one to deal also with morphisms. For Greechie diagrams there are no such results, and it seems highly problematic. Thus our new hypergraph representation improves the state of the art, and seems to capture the essence of the contextuality of orthoalgebras.

Projective geometry We may view hypergraphs of orthoalgebras as noncommutative analogues of projective geometries of vector spaces. The hypergraph of $\mathrm{BSub}(A)$ resembles the construction of a projective geometry out of the subspaces of a vector space V, where the bottom part of the lattice of subspaces $\mathrm{Sub}(V)$ forms a projective geometry that determines V. This similarity can be made more rigorous as follows. A Boolean algebra A is a vector space over $\mathbb{Z}_2$ under its symmetric difference operation. The subalgebras of A are the vector subspaces that contain 1 and are additionally closed under meet. Write $\langle 1 \rangle$ for the subspace generated by the vector 1, and consider the interval $I = [\langle 1 \rangle, A]$ of the subspace lattice. The subalgebras of height n in $\mathrm{BSub}(A)$ have height n in I. Since I is isomorphic to the subspace lattice of $A/\langle 1 \rangle$, we may regard $\mathrm{BSub}(A)$ as sitting inside a projective geometry over $\mathbb{Z}_2$. This explains the similarity of the hypergraph of a 16-element Boolean algebra to a Fano plane: the missing line is a subspace containing 1 that is not a subalgebra.

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