A universal completion of the ZX calculus

Kang Feng Ng Quanlong Wang

Department of Computer Science
University of Oxford
Oxford, UK

kangfeng.ng@cs.ox.ac.uk quanlong.wang@cs.ox.ac.uk

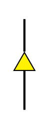
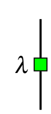
This extended abstract is based on the paper arXiv:1706.09877 [13].

The ZX calculus, introduced by Coecke and Duncan [4], is an intuitive and mathematically strict graphical language for quantum computing: it is formulated within the framework of compact closed categories which underpin the mathematics for graphical calculi [12]. The ZX calculus has diagrams and these diagrams have a standard interpretation in the Hilbert space, making it a convenient tool to represent quantum computing algorithms. Furthermore, it has intuitive and simple rewriting rules to transform diagrams to another, thus making it powerful tool for reasoning quantum algorithms diagrammatically. For the past ten years, the ZX calculus has enjoyed success in applying to fields of quantum information and quantum computation (QIC), in particular (topological) measurement-based quantum computing [6, 10] and quantum error correction [5, 3].

To realise the greatest advantage of the ZX calculus, the so-called completeness is of concern. Completeness of the calculus with respect to qubit quantum mechanics is the statement that any equation of diagrams that holds true under the standard interpretation in Hilbert space can be derived diagrammatically. It has been shown in [14] that the original ZX calculus [4] together with the Euler decomposition of the Hadamard gate is incomplete for the overall pure qubit quantum mechanics. Since then, plenty of efforts have been devoted to the completion of some part of quantum mechanics: real quantum mechanics [7], stabiliser quantum mechanics [1], single Clifford+T quantum mechanics [2], and Clifford+T quantum mechanics [11].

Amongst the list of completeness results, the completeness of ZX calculus for Clifford+T quantum mechanics is especially interesting, since it is approximately universal for quantum mechanics. This result depends on the completeness of another graphical calculus, the ZW calculus for "qubits with integer coefficients" [8]. We use a similar technique to prove a more general result, the completeness of the ZX calculus for the entire qubit quantum mechanics.

In this paper, we present a version of ZX calculus that is complete for the overall pure qubit quantum mechanics. In the process, we introduced two generators although they can be avoided in principle:

$T : 1 \to 1$	$L : 1 \to 1$
	

The rules of the calculus consist of two parts: the meta-rule where "only topology matters", and the rewriting rules shown in Figure 1. The first segment consists of the original ZX rules, the second segment

Submitted to:
QPL 2018

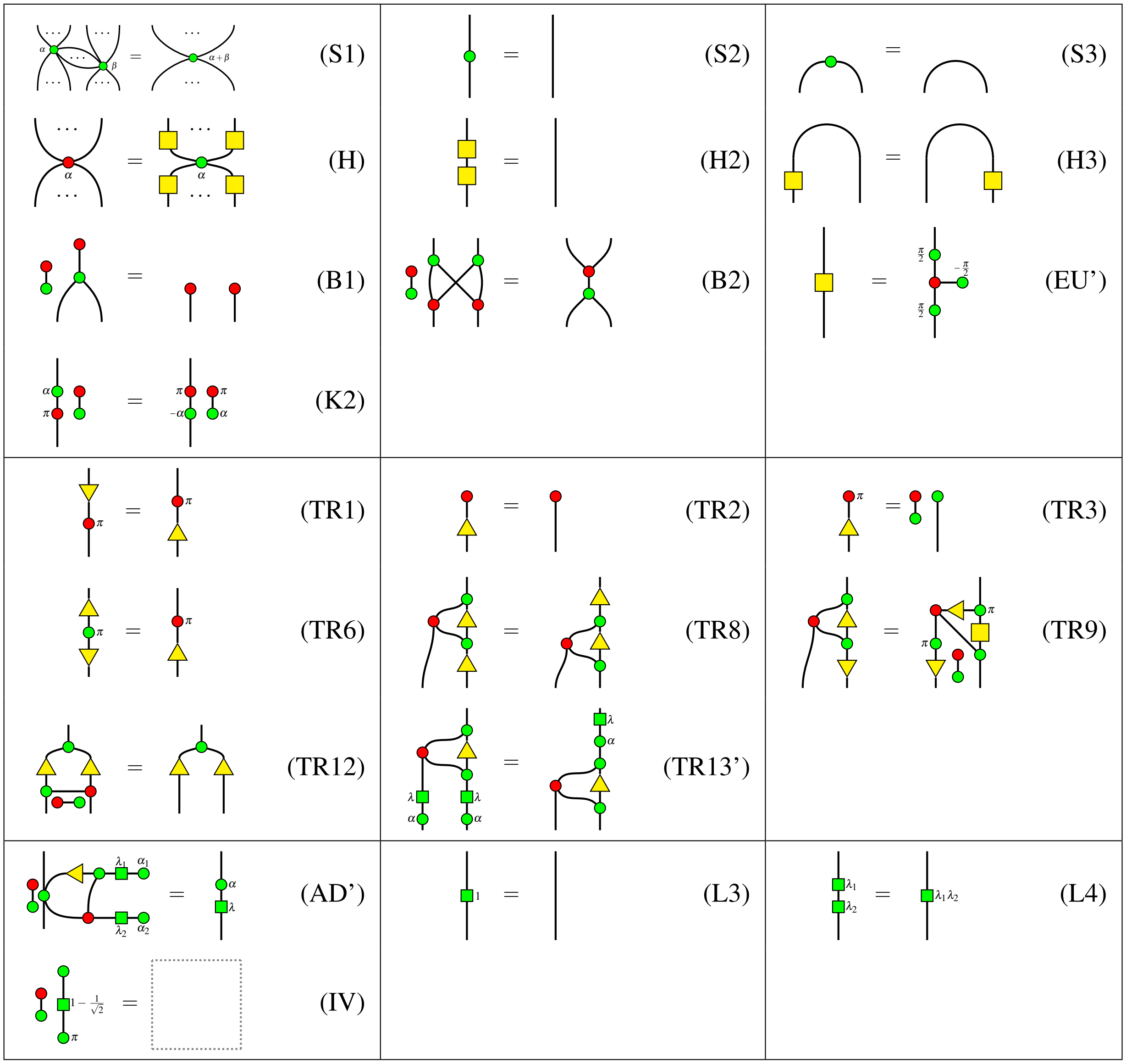


Figure 1: A set of ZX calculus rules. The new rules have an apostrophe on the rule names. $\alpha, \beta, \alpha_1, \alpha_2 \in [0, 2\pi)$, $\lambda, \lambda_1, \lambda_2 \geq 0$, and for rule (AD'), $\lambda e^{i\alpha} = \lambda_1 e^{i\alpha_1} + \lambda_2 e^{i\alpha_2}$.

are rules pertaining to the interaction of the triangle node with the other nodes, and the last segment are the rules for the λ box.

Note that we have updated the colour of the λ box (from yellow to green) and the rules that were first presented in [13].

Our proof of the completeness is based on the completeness of a newer ZW calculus for "qubits with coefficients over any commutative ring" [9], in particular over the complex numbers. We first establish a reversible translations from ZX to ZW calculus, vice versa, in the sense that we should be able to recover the same diagram after a couple of translations under the rules of the calculi. And by checking that all the rewriting rules in the ZW calculus are derivable under the translation from ZW to ZX calculus, we finally finished the proof of the completeness of the ZX calculus for the overall qubit quantum mechanics.

References

[1] M. Backens (2014): *The ZX-calculus is complete for stabilizer quantum mechanics*. New Journal of Physics 16(9), p. 093021, doi:10.1088/1367-2630/16/9/093021.

[2] M. Backens (2014): *The ZX-calculus is complete for the single-qubit Clifford+T group*. Electronic Proceedings in Theoretical Computer Science 172, pp. 293–303, doi:10.4204/eptcs.172.21.

[3] Nicholas Chancellor, Aleks Kissinger, Stefan Zohren & Dominic Horsman (2016): *Coherent parity check construction for quantum error correction*. arXiv preprint arXiv:1611.08012.

[4] B. Coecke & R. Duncan (2008): *Interacting quantum observables*. Automata, Languages and Programming, pp. 298–310.

[5] Ross Duncan & Maxime Lucas (2013): *Verifying the Steane code with Quantomatic*. Proceedings of the 10th International Workshop on Quantum Physics and Logic. arXiv:1306.4532.

[6] Ross Duncan & Simon Perdrix (2010): *Rewriting measurement-based quantum computations with generalised flow*. In: International Colloquium on Automata, Languages, and Programming, Springer, pp. 285–296.

[7] Ross Duncan & Simon Perdrix (2013): *Pivoting makes the ZX-calculus complete for real stabilizers*. Electronic Proceedings in Theoretical Computer Science.

[8] A. Hadzihasanovic (2015): *A Diagrammatic Axiomatisation for Qubit Entanglement*. In: Proceedings of the 2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science, IEEE Computer Society, pp. 573–584.

[9] A. Hadzihasanovic (2017): *The algebra of entanglement and the geometry of composition*. Ph.D. thesis, University of Oxford.

[10] Clare Horsman (2011): *Quantum picturalism for topological cluster-state computing*. New Journal of Physics 13(9), p. 095011.

[11] E. Jeandel, S. Perdrix & R. Vilmart (2017): *A Complete Axiomatisation of the ZX-Calculus for Clifford+ T Quantum Mechanics*. arXiv preprint arXiv:1705.11151.

[12] André Joyal & Ross Street (1991): *The geometry of tensor calculus, I*. Advances in mathematics 88(1), pp. 55–112.

[13] K. F. Ng & Q. Wang (2017): *A universal completion of the ZX-calculus*. arXiv preprint arXiv:1706.09877.

[14] Christian Schröder de Witt & Vladimir Zamdzhiev (2014): *The ZX-calculus is incomplete for quantum mechanics*. Electronic Proceedings in Theoretical Computer Science 172, pp. 285–292.