Agents, subsystems, and the conservation of information (extended abstract of [11])

Giulio Chiribella
giulio.chiribella@cs.ox.ac.uk
Department of Computer Science, University of Oxford, Oxford, UK
Canadian Institute for Advanced Research, CIFAR Program in Quantum Information Science, Toronto, Canada
Department of Computer Science, University of Hong Kong, Hong Kong, China
HKU Shenzhen Institute of Research and Innovation, Yuexing 2nd Rd Nanshan, Shenzhen 518057, China

The composition of physical systems is a fundamental structure in quantum foundations and in the study of general probabilistic theories. In this work we explore a different approach, where the basic structure is that of a single physical system. The system is then divided into subsystems, which are associated to submonoids of transformations, interpreted as the operations that can be performed by suitable agents. We then restrict our attention to systems where the physical transformations act invertibly, in agreement to what is sometimes called the Conservation of Information. For such systems, we propose a dynamical notion of pure states, and show that all the states of all subsystems admit a canonical purification. This result extends the purification principle to a broader setting, in which coherent superpositions can be interpreted as purifications of incoherent mixtures.

Compositional structures are ubiquitous in physics and computer science, They have been investigated in depth in quantum information theory [28, 23], and in the foundations of quantum mechanics, in particular in the framework of general probabilistic theories [18, 3, 6, 8, 5, 19, 21, 7, 12, 22, 27] and in categorical quantum mechanics [1, 15, 16, 2]. Mathematically, composition corresponds to the structure of monoidal category [16, 29], which can be informally regarded as a framework for constructing circuits. Larger systems are built in a bottom-up fashion, by combining subsystems into larger systems. When this is done, the overall system comes with a privileged decomposition into subsystems.

In this paper, we explore the opposite scenario. We start from an overall system, given without any preferred subdivision into subsystems. Then, we provide a construction that generates subsystems. Specifically, subsystems are identified by subsets of operations, which in principle can be performed by agents. This construction is closely related to an approach proposed by Krämer and del Rio, in which the states of a subsystem are identified with equivalence classes of states of the overall system [24]. In this paper, we extend this approach to transformations, providing a complete specification of the subsystems. We then restrict our attention to a special type of systems, satisfying the Conservation of Information [30], that is, the requirement that physical dynamics should send distinct states to distinct states. For systems satisfying the Conservation of Information, the future can be regarded as a faithful encoding of the past. Consistently with this picture, we put forward a dynamical notion of pure states, as the states that can be generated from a fixed initial state through the action of an invertible dynamics.

Having a flexible notion of subsystem is promising for the project of axiomatizing quantum theory [9, 20, 25, 17, 26, 31, 4]. First, axioms that refer to subsystems become more powerful when the notion of subsystem is broadened. Second, some of the physical principles assumed in the usual (compositional) framework may turn out to be consequences of the very definition of subsystem.

Submitted to:
QPL 2018

1 Agents and subsystems

Let S be a system, $\mathsf{St}(S)$ be its set of states, and $\mathsf{Transf}(S)$ be the set of transformations the system can undergo. We assume that $\mathsf{Transf}(S)$ is a monoid, with composition operation denoted by $\circ$ and identity element denoted by $\mathscr{I}_S$. We assume that there is an action of the monoid $\mathsf{Transf}(S)$ on the set $\mathsf{St}(S)$: given an input state $\psi \in \mathsf{St}(S)$ and a transformation $\mathscr{T} \in \mathsf{Transf}(S)$, the action of the transformation produces the output state $\mathscr{T}\psi \in \mathsf{St}(S)$.

An agent A is identified a set of actions, denoted as $\mathsf{Act}(A;S)$ and interpreted as the possible actions of A on S. Since the actions must be allowed physical processes, the inclusion $\mathsf{Act}(A;S) \subseteq \mathsf{Transf}(S)$ must hold. We also assume that $\mathsf{Act}(A;S)$ is a monoid, with the natural interpretation that executing one action right after another action is a possible action.

Definition 1. *Agents A and B* act independently *if the order in which they act is irrelevant, namely*

$$\mathscr{A} \circ \mathscr{B} = \mathscr{B} \circ \mathscr{A}\,, \qquad \forall \mathscr{A} \in \mathsf{Act}(A;S)\,, \mathscr{B} \in \mathsf{Act}(B;S)\,. \tag{1}$$

Intuitively, the above relation expresses the fact that A and B act on "different degrees of freedom" of the system and therefore their actions do not interfere with each other.

In an adversarial setting, agent B can be viewed as an adversary that tries to control as much of the system as possible.

We can think of the action of the adversary as a "degradation", and define the sets $\mathsf{Deg}(\psi) := \Big\{\mathscr{B}\psi\,, \quad \mathscr{B} \in \mathsf{Act}(B;S)\Big\}$ and $\mathsf{Deg}(\mathscr{T}) := \Big\{\mathscr{B}_1 \circ \mathscr{T} \circ \mathscr{B}_2\,, \quad \mathscr{B}_1, \mathscr{B}_2 \in \mathsf{Act}(B;S)\Big\}$.

Definition 2. *The* maximal adversary *of agent A is the agent A′ that can perform actions* $\mathsf{Act}\big(A';S\big) := \mathsf{Act}(A;S)'$.

Given an agent A, we define the subsystem S_A to be the collection of all degrees of freedom that are unaffected by the action of the maximal adversary A'. Consistently with this definition, we partition the states of S into disjoint subsets, so that each subset corresponds to a state of the subsystem S_A.

Following Krämer and del Rio [24], we define the equivalence relation $\phi \simeq_{A'} \psi$ iff there exists a finite sequence $(\psi_1, \psi_2, \dots, \psi_n)$ such that

$$\psi_1 = \phi\,, \qquad \psi_n = \psi\,, \qquad \text{and} \qquad \mathsf{Deg}(\psi_i) \cap \mathsf{Deg}(\psi_{i+1}) \neq \emptyset \quad \forall i \in \{1,2,\dots,n-1\}\,. \tag{2}$$

We then define the states of system S_A as equivalence classes with respect to this relation.

The transformations on system S_A can be constructed with an equivalence class procedure similar to the one used for states. But before taking equivalence classes, we need a candidate set of transformations acting exclusively on A's degrees of freedom. The largest candidate set is the set of all transformations that commute with the actions of the maximal adversary A', namely $\mathsf{Act}(A;S)''$. In general $\mathsf{Act}(A;S)''$ could be larger than $\mathsf{Act}(A;S)$, in agreement with the fact the set of physical transformations of system S_A could be larger than the set of operations that agent A can perform.

We then define $\mathscr{S} \simeq_{A'} \mathscr{T}$ iff there exists a finite sequence $(\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_n)$ such that

$$\mathscr{A}_1 = \mathscr{S}\,, \qquad \mathscr{A}_n = \mathscr{T}\,, \qquad \text{and} \qquad \mathsf{Deg}(\mathscr{A}_i) \cap \mathsf{Deg}(\mathscr{A}_{i+1}) \neq \emptyset \quad \forall i \in \{1,2,\dots,n-1\}\,. \tag{3}$$

In analogy with the case of states, we denote the above relation as $\mathscr{S} \simeq_{A'} \mathscr{T}$, and we observe that $\rightsquigarrow_{A'}$ is an equivalence relation. We denote the equivalence class of the transformation $\mathscr{T}$ by $[\mathscr{T}]_{A'}$, and we stipulate that the transformations of subsystem S_A are the equivalence classes. The composition of two transformations $[\mathscr{A}_1]_A$ and $[\mathscr{A}_2]_A$ is defined in the obvious way, namely

$$[\mathscr{A}_1]_A \circ [\mathscr{A}_2]_A := [\mathscr{A}_1 \circ \mathscr{A}_2]_A\,. \tag{4}$$

Similarly, the action of the transformations on the states is as

$$[\mathscr{A}]_A\,[\psi]_A := [\mathscr{A}\,\psi]_A\,. \tag{5}$$

Definitions (4) and (5) are well-posed, in the sense that their right hand sides are independent of the choice of representatives for the equivalence classes.

2 The Conservation of Information

We now consider physical systems where all transformations are *physically invertible*:

Definition 3. *A transformation* $\mathscr{T} \in \mathsf{Transf}(S)$ *is* physically invertible *iff there exists another transformation* $\mathscr{T}' \in \mathsf{Transf}(S)$ *such that* $\mathscr{T}' \circ \mathscr{T} = \mathscr{I}_S$.

We call the requirement that all transformations be invertible the Physical Conservation of Information. Note that, if the system S satisfies the Physical Conservation of Information, then the monoid $\mathsf{Transf}(S)$ is actually a group, hereafer denoted as $\mathsf{G}(S)$.

Now, imagine that an agent A acts on a system S satisfying the Physical Conservation of Information. The set of actions of agent A contains all the inverses of its elements, and therefore is itself a group, denoted as G_A. The most powerful adversary of A is the adversary $B = A'$, who has access to all transformations in the set $\mathsf{G}_B := \mathsf{G}_A'$. It is immediate to see that the set G_B is a group, hereafter called the *adversarial group*.

We show that, under certain assumptions, physical systems satisfying the Conservation of Information also satisfy the purification property [8, 10, 14, 7, 12, 13, 27], namely the property that every mixed state can be modelled as a pure state of a larger system in a canonical way. In order to formulate the notion of purification in the context of this paper, we first need a suitable definition of pure states.

We explore a different definition, inspired by the dynamicist approach adopted in this paper. The key notion here is the notion of *cyclic state*, namely a fixed state from which all the others can be generated:

Definition 4. *The state* $\psi_0 \in \mathsf{St}(S)$ *is* cyclic *iff for every other state* $\psi \in \mathsf{St}(S)$ *there exists a transformation* $\mathscr{T} \in \mathsf{Transf}(S)$ *such that* $\psi = \mathscr{T}\,\psi_0$.

The ability to generate every state from a fixed state ψ_0 is important *per se*, and can be regarded as a *desideratum* of our physical theory:

Axiom 1 (Initialization)**.** *There exists at least a cyclic state in* $\mathsf{St}(S)$.

The Initialization Axiom holds in quantum theory, both in the pure state version and in the mixed state version. In the pure state version, every unit vector $|\psi\rangle \in \mathscr{H}_S$ can be generated from a fixed unit vector $|\psi_0\rangle \in \mathscr{H}_S$ via a unitary transformation U. In the mixed state version, every density matrix ρ can be generated from a fixed density matrix ρ_0 via the quantum channel $\mathscr{C}_\rho = \rho\ \mathrm{Tr}$.

To capture the notion of pure state, we combine the Initialization Axiom and the Conservation of Information:

Definition 5 (Dynamicist definition of pure states)**.** *If system S satisfies both the Physical Conservation of Information and the Initialization Axiom, then the states of system S are called* pure.

When the dynamical definition of pure states can be adopted, we have the following

Proposition 1 (Purification)**.** *Let S be a system satisfying the Physical Conservation of Information and the Initialization Axiom. Let A be an agent in S, and let* $B = A'$ *be its maximal adversary. Then, for every state* $\rho \in \mathsf{St}(S_A)$*, there exists a pure state* $\psi \in \mathsf{St}(S)$*, such that* $\rho = \mathrm{Tr}_B[\psi]$*. Moreover, if* $\psi' \in \mathsf{St}(S)$ *is another pure state with* $\mathrm{Tr}_B[\psi] = \rho$*, then there exists a reversible transformation* $\mathscr{U}_B \in \mathsf{G}_B$ *such that* $\psi' = \mathscr{U}_B \psi$.

3 Conclusions and outlook

In this paper we adopted rather minimalistic framework, in which a *single* physical system was described solely in terms of states and transformations. It is interesting to see how the definition of subsystem adopted in this paper interacts with probabilities. A natural question is: under which condition can one retrieve the convexity of all state spaces? Another interesting direction of future research is to enrich the structure of system with additional features, such as a metric quantifying the proximity of states. In particular, one may consider a strengthened formulation of the Conservation of Information, in which the physical transformations are required not only to be invertible, but also to preserve the distances. It is then interesting to consider how the metric on the pure states of the whole system induces a metric on the subsystems, and to search for relations between global metric and local metric. These potential avenues of future research suggest that the notions investigated in this work may find application in a variety of different contexts, and for a variety of interpretational standpoints.

Acknowledgements. This work is supported by the Foundational Questions Institute through grant FQXi-RFP3-1325, the National Natural Science Foundation of China through grant 11675136, the Croucher Foundation, the Canadian Institute for Advanced Research (CIFAR), and the Hong Research Grant Council through grant 17326616. This publication was made possible through the support of a grant from the John Templeton Foundation. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

References

[1] Samson Abramsky & Bob Coecke (2004): *A categorical semantics of quantum protocols*. In: *Logic in Computer Science, 2004. Proceedings of the 19th Annual IEEE Symposium on*, IEEE, pp. 415–425.

[2] Samson Abramsky & Bob Coecke (2008): *Categorical quantum mechanics*. *Handbook of quantum logic and quantum structures: quantum logic*, pp. 261–324.

[3] Howard Barnum, Jonathan Barrett, Matthew Leifer & Alexander Wilce (2007): *Generalized no-broadcasting theorem*. *Physical review letters* 99(24), p. 240501.

[4] Howard Barnum, Markus P Müller & Cozmin Ududec (2014): *Higher-order interference and single-system postulates characterizing quantum theory*. *New Journal of Physics* 16(12), p. 123029.

[5] Howard Barnum & Alexander Wilce (2011): *Information processing in convex operational theories*. *Electronic Notes in Theoretical Computer Science* 270(1), pp. 3–15.

[6] Jonathan Barrett (2007): *Information processing in generalized probabilistic theories*. *Physical Review A* 75(3), p. 032304.

[7] G. Chiribella (2014): *Dilation of states and processes in operational-probabilistic theories*. In B. Coecke, I. Hasuo & P. Panangaden, editors: Proceedings 11th workshop on *Quantum Physics and Logic*, Kyoto, Japan, 4-6th June 2014, *Electronic Proceedings in Theoretical Computer Science* 172, pp. 1–14, doi:10.4204/EPTCS.172.1.

[8] G. Chiribella, G.M. D'Ariano & P. Perinotti (2010): *Probabilistic theories with purification*. *Phys. Rev. A* 81, p. 062348.

[9] G. Chiribella, G.M. D'Ariano & P. Perinotti (2011): *Informational derivation of quantum theory*. *Phys. Rev. A* 84, p. 012311.

[10] G. Chiribella, G.M. D'Ariano & P. Perinotti (2012): *Quantum Theory, namely the pure and reversible theory of information*. *Entropy* 14(10), pp. 1877–1893.

[11] Giulio Chiribella (2018): *Agents, Subsystems, and the Conservation of Information*. *arXiv preprint arXiv:1511.08818*.

[12] Giulio Chiribella, Giacomo Mauro D?Ariano & Paolo Perinotti (2016): *Quantum from principles*. In: Quantum Theory: Informational Foundations and Foils, Springer, pp. 171–221.

[13] Giulio Chiribella & Carlo Maria Scandolo (2015): *Conservation of information and the foundations of quantum mechanics*. In: EPJ Web of Conferences, 95, EDP Sciences, p. 03003.

[14] Giulio Chiribella & Xiao Yuan (2013): *Quantum theory from quantum information: the purification route*. Canadian Journal of Physics 91(6), pp. 475–478.

[15] B. Coecke (2006): *Kindergarten quantum mechanics: lecture notes*. In: Quantum Theory: Reconsideration of Foundations-3, AIP Conf. Proc. **810**, American Institute of Physics Melville, NY, pp. 81–98.

[16] Bob Coecke (2010): *Quantum picturalism*. Contemporary physics 51(1), pp. 59–83.

[17] B. Dakic & C. Brukner (2011): *Quantum Theory and Beyond: Is Entanglement Special?* In H. Halvorson, editor: Deep Beauty: Understanding the Quantum World through Mathematical Innovation, Cambridge University Press, pp. 365–392.

[18] Lucien Hardy (2001): *Quantum theory from five reasonable axioms*. arXiv preprint quant-ph/0101012.

[19] Lucien Hardy (2011): *Foliable operational structures for general probabilistic theories*. Deep Beauty: Understanding the Quantum World through Mathematical Innovation; Halvorson, H., Ed, p. 409.

[20] Lucien Hardy (2011): *Reformulating and reconstructing quantum theory*. arXiv preprint arXiv:1104.2066.

[21] Lucien Hardy (2013): *A formalism-local framework for general probabilistic theories, including quantum theory*. Mathematical Structures in Computer Science 23(02), pp. 399–440.

[22] Lucien Hardy (2016): *Reconstructing quantum theory*. In: Quantum Theory: Informational Foundations and Foils, Springer, pp. 223–248.

[23] Alexei Yu Kitaev, Alexander Shen & Mikhail N Vyalyi (2002): *Classical and quantum computation*. 47, American Mathematical Soc.

[24] Lea Kraemer & Lídia Del Rio (2017): *Operational locality in global theories*. arXiv preprint arXiv:1701.03280.

[25] L. Masanes & M. P. Müller (2011): *A derivation of quantum theory from physical requirements*. New Journal of Physics 13(6):063001, doi:10.1088/1367-2630/13/6/063001.

[26] L. Masanes, M. P. Müller, R. Augusiak & D. Perez-Garcia (2013): *Existence of an information unit as a postulate of quantum theory*. Proceedings of the National Academy of Science 110, pp. 16373–16377, doi:10.1073/pnas.1304884110.

[27] Giacomo Mauro D'Ariano, Giulio Chiribella & Paolo Perinotti (2017): *Quantum Theory from First Principles*. Quantum Theory from First Principles, by Giacomo Mauro D'Ariano, Giulio Chiribella, Paolo Perinotti, Cambridge, UK: Cambridge University Press, 2017.

[28] Michael Nielsen & I Chuang (2000): *Quantum information and computation*. Quantum Information and Computation.

[29] Peter Selinger (2011): *A survey of graphical languages for monoidal categories*. In: New structures for physics, Springer, pp. 289–355.

[30] Leonard Susskind (2008): *The black hole war: My battle with Stephen Hawking to make the world safe for quantum mechanics*. Hachette UK.

[31] A. Wilce (2012): *Conjugates, Filters and Quantum Mechanics*.