Reconstructing quantum theory from diagrammatic postulates

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Abstract

The following is an extended abstract for [Selby, John H., Carlo Maria Scandolo, and Bob Coecke. *"Reconstructing quantum theory from diagrammatic postulates."* arXiv:1802.00367 (2018)].

1 Introduction

A reconstruction of quantum theory refers to both a mathematical and a conceptual paradigm that allows one to derive the usual formulation of quantum theory from a set of primitive assumptions. In our case this mathematical paradigm is the diagrammatic calculus afforded to symmetric monoidal categories, whilst the conceptual paradigm is the interpretation of said diagrams as diagrams of physical processes.

The motivation for reconstructing quantum theory in such a way, is a discomfort with the usual formulation of quantum theory, a discomfort that started with its originator John von Neumann. Merely three years after the publication of his book [14] that cemented the mathematical formalism of quantum theory, he made it clear in a letter to the mathematician Garrett Birkhoff that he was no longer satisfied with the Hilbert space formalism [8]. A more modern perspective is that reconstructions help to see the physical principles that underlie the Hilbert space structure. This in turn allows one both to derive more applications of quantum theory, and to understand how it can, and should, be modified in order to reconcile it with general relativity.

2 Main result

In [10] we present a reconstruction of finite-dimensional quantum theory where all of the postulates are stated entirely in diagrammatic terms, making them intuitive. Equivalently, they are stated in category-theoretic terms, making them mathematically appealing. Again equivalently, they are stated in process-theoretic terms, establishing the fact that the conceptual bare-bones of quantum theory concerns the manner in which systems and processes compose.

The paper provides a reconstruction that is conceptually grounded whilst still being based on crisp mathematical axioms. This is achieved by exploiting the correspondence:

$$\text{diagrams} \simeq \text{category theory} \simeq \text{process theory}$$

Our postulates are now entirely diagrammatic, i.e. entirely category-theoretic, i.e. entirely process-theoretic. This provides them with an intuitive, elegant as well as principled underpinning. In short, what we prove is:

$$\text{classical interface} + \text{postulates about how processes compose} \implies \text{quantum processes}$$

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More explicitly, the complete list of postulates that we use to reconstruct quantum theory are as follows:

1. the theory is a process theory,
2. with a finite local classical interface,
3. cups and caps,
4. a sharp-dagger,
5. and in which all processes admit symmetric purifications.

These principles are formally defined in [10]. Conceptually, they may be interpreted as follows:

- That the theory is a process theories states that nature is fundamentally about processes and how they compose.
- The classical interface describes how we interact with, control, and learn about the theory.
- Cups and caps represent a fundamental symmetry which ensures that the theory has maximal correlations.
- The sharp dagger represent another fundamental symmetry, namely time reversal, in which the time reverse of a state corresponds to a 'test' for that state.
- Finally, the existence of symmetric purifications ensures that any lack of purity can always be traced to lack of access to the past or future of certain systems.

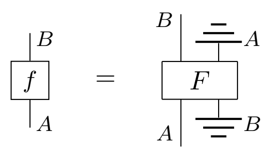
The proof of this result is remarkably simple in contrast to many other reconstructions, largely owing to the use of the diagrammatic proofs. Moreover the structure of the reconstruction is particularly clear and can be summarised in a simple flowchart [10, Fig. 1]. Aside from the diagrammatic proofs, this simplicity can be traced to the use of the Koecher-Vinberg theorem [7, 13]. — borrowed from the works of Barnum, Wilce et al., for example [3, 4, 15].

In the following section we elaborate on the most novel and conceptually interesting of our postulates, the symmetric purification postulate. In the final section we will discuss how this approach relates to other frameworks for reconstructing quantum theory and to the research programme of categorical quantum mechanics.

3 Symmetric purification

Aside from the diagrammatic form, a key aspect of this reconstruction is the introduction of a new postulate: *symmetric purification*. Symmetric purification expresses the requirement that all processes of the theory are fundamentally pure, and any apparent lack of purity should arise from lacking information about and control over the past and/or future of some environment systems, formally:

Postulate : Symmetric purification. *Every process $f : A \to B$ can be dilated to a* pure *process $F : A \otimes B \to B \otimes A$ as follows:*



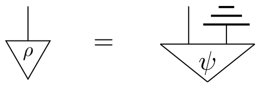
where we call F a purification *of f. Moreover, purifications are* essentially unique.

Unlike the ordinary purification postulate, symmetric purification applies equally well to classical theory as well as quantum theory. We therefore first reconstruct the full process theoretic description of quantum theory, consisting of hybrid classical-quantum systems and their interactions, before restricting ourselves to just the 'fully quantum' systems in a final step.

We propose two novel alternative manners of doing so, 'no-leaking' (roughly that information gain causes disturbance) and 'purity of cups' (roughly the existence of entangled states). Whilst at first glance these appear to be unrelated postulates involving different diagrammatic concepts, they can actually be shown to be equivalent in any process theory with cups and caps.

The standard purification postulate was first used in [5] as an operational generalisation of the Stinespring dilation theorem [12]. It roughly states that any mixed process can be represented as a pure process with an extra output that is discarded, formally:

Postulate : Purification. *Every state* $\rho : I \to A$ *can be dilated to a* pure *state* $\psi : I \to A \otimes B$ *as follows:*



where we call ψ *a* purification *of* ρ*. Moreover, purifications are* essentially unique *[6].*

As mentioned before, this postulate is satisfied by quantum theory but not by classical probability theory, so in the reconstruction of [5] it is used to single out quantum theory. We show however that the standard purification postulate is implied by the conjunction of the symmetric purification postulate and the existence of a *pure cup*. Our work can therefore be seen as deconstructing the standard purification postulate into two parts —one which applies to both quantum and classical theory, and the other only to quantum— therefore refining exactly what it is that is uniquely quantum about the postulate.

4 Connection to GPTs, OPTs and CQM

Other tangential results concern the specific frameworks of generalised probabilistic theories (GPTs) and process theories (a.k.a. CQM). While initially GPTs only appealed to single systems, under the impetus of CQM a new hybrid form was proposed (a.k.a. OPTs), where multiple systems are supported by means of a diagrammatic backbone. We demonstrate that, within the context of process theories, the essential structure of OPTs can be derived from the classical interface postulate. Hence, GPTs and OPTs can be subsumed within the process theory framework.

Secondly, we have now characterised necessary additional axioms for a process theory to correspond to the Hilbert space model, and in particular, that a 'sharp dagger' is indeed the right choice of a dagger structure. Cups and caps have been part of the structure of CQM from the start [1] and provide a cup- and a cap-shaped wire for each system: allowing for inputs to be connected to inputs and outputs to outputs. Daggers have also been part of CQM since its very beginning [2, 11]. Unlike the transpose, which is constructed using cups and caps, a dagger did not have any other structural requirements besides being compositional. In order to fully characterise the Hermitian adjoint of quantum theory, a *sharpness* constraint was added in [9]. We then demonstrate that once we have a classical interface, cups and caps, and a sharp dagger, that the only additional postulate to be imposed, for a process theory to correspond to the Hilbert space model, is the aforementioned symmetric purification postulate.

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