Limits in dagger categories (extended abstract)

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Introduction Dagger categories occur naturally as models of quantum theory [3]. A dagger on a category is a contravariant functor $(-)^\dagger$ satisfying $A^\dagger = A$ on objects and $f^{\dagger\dagger} = f$ on morphisms. In the category of Hilbert spaces, the role of a dagger is played by adjoints. The presence of a dagger seems to model the fact that the laws of nature are invariant under time reversal, or at least a principle of preservation of information. But so far there has been no convincing physical interpretation of the dagger. Nevertheless, the dagger is undeniably a powerful ingredient in many categorical approaches to quantum theory, including approaches that are unwittingly categorical in spirit but not explicitly so.

In this work, we study *dagger category theory* in its own right. The hope is that by elucidating the categorical behaviour of the dagger, we can eventually identify its physical significance. For now we stick to purely mathematical results, without an attempt at physical interpretation [4].

Dagger limits Dagger category theory differs from category theory in two essential ways. First, dagger categories are *self-dual* in a strong sense. Consequently dagger categories behave differently than categories on a fundamental level. Categories are intuitively built from objects $\bullet$ and morphisms $\bullet \to \bullet$. Dagger categories are built from $\bullet$ and $\bullet \leftrightarrows \bullet$. The result is a theory that is essentially directionless.

Second, objects in a dagger category do not behave the same when they are merely isomorphic, but only when the isomorphism respects the dagger. A case in point is the dagger category of Hilbert spaces and continuous linear functions. Two objects are isomorphic when there is a linear homeomorphism with respect to the two topologies induced by the inner products. The inner product itself need not be respected by the isomorphism, unless it is *unitary*.

We study *limits* in dagger categories, bringing these two features of self-duality and unitarity to the forefront. If $l_A\colon L \to D(A)$ is a limit for a diagram $D\colon \mathbf{J} \to \mathbf{C}$, then $l_A^\dagger\colon D(A) \to L$ is a colimit for $\dagger \circ D\colon \mathbf{J}^{\mathrm{op}} \to \mathbf{C}$, so L has two universal properties; they should be compatible with each other. Moreover, a *dagger limit* should be unique not just up to mere isomorphism but up to unitary isomorphism. Our main contribution is the following definition.

Definition 3.1. *Let* $\mathbf{C}$ *be a dagger category and* $\mathbf{J}$ *a category. A class* Ω *of objects of* $\mathbf{J}$ *is* weakly initial *if for every object* B *of* $\mathbf{J}$ *there is a morphism* $f\colon A \to B$ *with* $A \in \Omega$, *i.e. if every object of* $\mathbf{J}$ *can be reached from* Ω. *Let* $D\colon \mathbf{J} \to \mathbf{C}$ *be a diagram and let* $\Omega \subseteq J$ *be weakly initial.* A dagger limit of D with support Ω *is a limit* L *of* D *whose cone* $l_A\colon L \to D(A)$ *satisfies the following two properties:*

normalization *l_A is a partial isometry for every $A \in \Omega$, i.e.* $l_A l_A^\dagger l_A = l_A$;

independence *the projections on L induced by these partial isometries commute, i.e.* $l_A^\dagger l_A l_B^\dagger l_B = l_B^\dagger l_B l_A^\dagger l_A$ *for all* $A, B \in \Omega$.

A dagger limit of D is a dagger limit with support some weakly initial class. If L is a dagger limit of D, we will also write $L = \mathrm{dlim} D$.

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Related work The rest of this extended abstract highlights evidence for Definition 3.1 being the 'correct' definition of dagger limits. First of all, it subsumes all known examples from the literature:

- A *dagger (bi)product* [3] is a dagger limit where $\mathbf{J}$ is the discrete category on two objects that D sends to A and B, and Ω necessarily consists of both objects.
- A *dagger equalizer* [5] is a dagger limit where $\mathbf{J} = \bullet \rightrightarrows \bullet$ and Ω consists of only the first object. This in particular includes *dagger kernels* [1].
- A *dagger intersection* [5] is a dagger limit of a diagram of isometries into a common object.
- A *dagger splitting* of a dagger idempotent [4] is a dagger limit of the diagram generated by it.

Moreover, we prove that $\mathbf{Rel}$, $\mathbf{PInj}$, and $\mathbf{Span}(\mathbf{FinSet})$ have dagger limits for all connected dagger categories $\mathbf{J}$, and that the limit-colimit coincidence from domain theory is an instance of a dagger limit.

However, in contrast to earlier work [5], notice that Definition 3.1 needs no enrichment of any sort, and applies to arbitrary (infinite) diagrams.

Uniqueness The support Ω is important because of the following theorem.

Theorem 4.1. *Let L and M be dagger limits of $D\colon \mathbf{J} \to \mathbf{C}$. Then L and M are unitarily isomorphic as limits if and only if they are both dagger limits with support in the same weakly initial class.*

Completeness Perhaps the most obvious definition of dagger completeness would be "every diagram has a dagger limit for some/all possible supporting subsets". However, such a definition would be too strong to allow interesting models: we prove that if a dagger category has dagger equalizers, dagger pullbacks and finite dagger products, then it must be indiscrete; and similarly that if a dagger category has dagger equalizers and infinite dagger products, then it must be indiscrete. Instead, we propose that dagger completeness should mean having dagger limits for a certain class of shapes $\mathbf{J}$, that contains equalizers, arbitrary intersections, and finite products.

Definition 5.5. *A class Ω of objects of a category $\mathbf{J}$ is called a* basis *when every object B allows a unique $A \in \Omega$ making $\mathbf{J}(A,B)$ non-empty. The category $\mathbf{J}$ is called* based *when there exists a basis, and* finitely based *when there exists a finite basis. We say a dagger category $\mathbf{C}$ has* (finitely) based dagger limits*, or that it is* (finitely) based dagger complete *if for every category $\mathbf{J}$ with a (finite) basis Ω, any diagram $D\colon \mathbf{J} \to \mathbf{C}$ has a dagger limit with support Ω. In the context of a diagram $D\colon \mathbf{J} \to \mathbf{C}$, we will also call $\mathbf{J}$ the* shape *of the diagram.*

Notice that finitely based diagrams need not be finite.

Theorem 5.8. *A dagger category is finitely based dagger complete if and only if it has dagger equalizers, dagger intersections and finite dagger products.*

Theorem 5.9. *Let κ be a cardinal number. A dagger category has dagger-shaped limits of shapes with at most κ many connected components if and only if it has dagger split infima of projections, dagger stabilizers, and dagger products of at most κ many objects.*

We also prove (in Proposition 5.12) that when $\mathbf{J}$ has a finite basis and $\mathbf{C}$ is appropriately enriched, Definition 5.5 coincides with the notion of completeness in [5].

Adjoints If limits exist for all diagrams of a fixed shape, it is well known that they can also be formulated as an adjoint to the constant functor. Definition 3.1 makes a dagger version of this theorem true. In Theorems 6.1–6.3, we prove that this is the case under reasonable assumptions, such as the adjunction being a dagger adjunction [2], and the counit being a partial isometry when restricted to Ω.

We also explore dagger versions of adjoint functor theorems (in Section 7). There are obstacles to such theorems: the solution set condition only concerns single objects, whereas a dagger adjoint functor theorem needs adjointability of maps between objects. We prove this weak version in Theorem 7.1.

Polar decomposition Polar decomposition provides a way to factor any bounded linear map between Hilbert spaces into a partial isometry and a positive morphism. We prove that the presence of polar decomposition turns limits into dagger limits. Identifying the abstract property of polar decomposition categorically is of independent interest, especially for quantum foundations. Recall that a bimorphism is a morphism that is both an epimorphism and a monomorphism.

Definition 8.1. *Let $f\colon A \to B$ be a morphism in a dagger category. A* polar decomposition *of f consists of two factorizations of f as $f = pi = jp$,*

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & A \\ {\scriptstyle p}\big\downarrow & \searrow^{f} & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[\;j\;]{} & B \end{array}$$

where p is a partial isometry and i and j are self-adjoint bimorphisms.

For example, it follows (Proposition 8.3) that in dagger categories with polar decomposition, objects that are isomorphic are in fact also unitarily isomorphic.

Theorem 8.5. *Let Ω be a basis of $\mathbf{J}$. Assume that $D\colon \mathbf{J} \to \mathbf{C}$ has a limit $l_A\colon L \to D(A)$ satisfying $l_A^\dagger l_A l_B^\dagger l_B = l_B^\dagger l_B l_A^\dagger l_A$ for $A, B \in \Omega$. If $\mathbf{C}$ admits polar decomposition, D has a dagger limit with support Ω.*

Commutativity Finally, we prove that dagger limits commute with dagger (co)limits under reasonable assumptions. This is strong behaviour when compared to ordinary categories, where limits do not often commute with colimits.

Theorem 9.5. *If $\mathbf{C}$ has all $\mathbf{J}$-shaped dagger limits with support Ω_J and all $\mathbf{K}$-shaped dagger colimits with support Ω_K, and $D\colon \mathbf{J} \times \mathbf{K} \to \mathbf{C}$ is an adjointable bifunctor, then the canonical morphism $\mathrm{dcolim}_k \,\mathrm{dlim}_j D(j,k) \to \mathrm{dlim}_j \,\mathrm{dcolim}_k D(j,k)$ is unitary.*

References

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