Pure maps between Euclidean Jordan Algebras

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We propose a definition of purity for positive linear maps between Euclidean Jordan Algebras (EJA) that generalizes the notion of Kraus rank one channels ($\mathrm{ad}_E \equiv E^*(\,\cdot\,)E$). We show that this definition of purity is closed under composition and taking adjoints and thus that the pure maps form a dagger category. This dagger yields a notion of positivity: maps of the form $f \equiv g^\dagger \circ g$ are called †-positive. We show that such a †-positive map f is completely determined by $f(1)$ being equal to $Q_{\sqrt{f(1)}}$, the Jordan algebraic version of the sequential measurement map $\mathrm{ad}_{\sqrt{f(1)}}$. The notion of †-positivity therefore characterises the sequential product.

These results hold for the general reason that the opposite category of EJAs with positive contractive linear maps is a †-effectus. The notion of †-effectus was introduced to prove similar results for the opposite category of normal completely positive contractive maps between von Neumann algebras.

1 Introduction

The concept of purity has proven very useful in the field of quantum information. In the context of states, it can be considered a resource in various protocols and computations [10, 3]. The possibility of purification of states is considered to be one of the characteristic features differentiating quantum theory from its classical counterpart [4, 5]. While there is a generally accepted definition of purity for states, when it comes to quantum channels there are several definitions of purity in play. There is for instance the definition of *atomicity* used in reconstructions of quantum theory [5], purity defined in terms of leaks [20], using orthogonal factorization [9] and purity defined using filters and corners [25, 24]. In this paper we use this last definition of purity. It was originally defined and studied for von Neumann algebras [23], but has since been studied in the abstract setting of a †-*effectus* [25].

A commonly used technique when studying the foundations of quantum theory, is to consider generalised theories that only exhibit some part of the features of conventional quantum theory. In this way, it becomes more clear what specific properties of quantum theory lead to certain structure. One of the first generalised quantum theories to be studied are the Euclidean Jordan algebras (EJAs) [14]. Besides the matrix algebras of complex self-adjoint matrices of conventional quantum theory, other examples of EJAs are the set of real symmetric matrices of real-valued quantum theory, or the set of self-adjoint matrices over the quaternions. Quite soon after the introduction of EJAs, a full characterisation of EJAs was given [15] that showed that these examples almost completely exhaust the possibilities. The Koecher–Vinberg theorem [18] is a major result that states that any ordered vector space with a homogeneous self-dual positive cone is a Euclidean Jordan algebra. It is this theorem that explains the ubiquity of EJAs in reconstructions of quantum theory [2, 26, 19].

In this paper we show that Euclidean Jordan algebras allow a definition of purity that generalises the notion of purity for (finite-dimensional) von Neumann algebras defined in [25, 24]. This notion of purity involves some concepts that have not been studied before in the context of EJAs and that might be of independent interest. For instance we show that the category $\mathbf{EJA}_{psu}$ of EJAs with positive

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subunital maps between them has *filters and corners*. These maps were introduced as *quotients and comprehensions* in [6] and they are pure maps which have a certain universal property. Their existence in von Neumann algebras was shown in [22]. To show that our definition of purity is closed under composition we need to show that the filters and corners are suitably related, for which we need to prove a generalisation of the *polar decomposition* present in von Neumann algebras.

Since EJAs are Hilbert spaces, every bounded map allows an adjoint. We show that the adjoint of a pure map is again pure so that the pure maps form a dagger category. This is in contrast to the other notions of purity considered before, where the pure maps are in general not closed under adjoints, or where the identity map is not always pure.

We also undertake a study of the *possibilistic* structure of the maps between EJAs. Taking this viewpoint we do not look at probabilities, but instead only consider when a probability is nonzero. By defining a notion of $\diamond$-*adjointness* we can also define $\diamond$-self-adjointness and $\diamond$-positivity. We completely characterise the pure $\diamond$-positive maps and show that they exactly correspond to a generalisation of the sequential product maps $b \mapsto \sqrt{a}b\sqrt{a}$ in von Neumann algebras. This result can be seen as a characterisation of the sequential product like the ones given in [12, 27, 22].

Our results can be summarised by saying that the category $\mathbf{EJA}_{psu}$ is a †-effectus [25]. In a previous paper the third author has shown that any *operational* †-effectus is a subcategory of $\mathbf{EJA}_{psu}$. This paper can therefore be seen as proving the converse result.

In the next section we review some of the basic structure present in Euclidean Jordan algebras. In section 3 we introduce the notion of *filters* and *corners* and study our notion of purity. Finally in section 4 we use the possibilistic $\diamond$ structure to characterise the sequential product maps as the unique pure $\diamond$-positive maps.

2 Preliminaries

We begin by giving a definition of Euclidean Jordan algebras and some motivating examples. Afterwards we will review some of the basic theory necessary to develop our results.

Definition 1. A *Jordan algebra* $(E,*,1)$ is a real unital commutative (possibly non-associative) algebra satisfying the *Jordan identity*: $(a*b)*(a*a) = a*(b*(a*a))$. A *Euclidean Jordon algebra* E is a Jordan algebra endowed with an inner product $\langle\cdot,\cdot\rangle$ which turns E into a real Hilbert space with $\langle a*b,c\rangle = \langle b,a*c\rangle$ for all $a,b,c\in E$,

Note 2. In the original (and frequently used) definition, one additionally requires a Euclidean Jordan algebra to be finite dimensional. The possibly infinite-dimensional version we use is also called a JH-algebra [8].

Example 3. Let F be the field of real numbers, the field of complex numbers, or the division algebra of the quaternions. Let $A \in M_n(F)$ be an $n\times n$ matrix over F. We call A *self-adjoint* when $A_{ij} = \overline{A_{ji}}$ where $\overline{\lambda}$ denotes the standard involution on F. We let the set of self-adjoint matrices be denoted by $M_n(F)^{\mathrm{sa}}$. This set is an Euclidean Jordan Algebra with the Jordan product $A*B := \frac{1}{2}(AB+BA)$, inner product $\langle A,B\rangle := \mathrm{tr}(AB)$ and identity matrix as unit.

If $n=3$ and F is the algebra of octonions, then the algebra $M_3(F)^{\mathrm{sa}}$ is also a Euclidean Jordan algebra, which is called *exceptional*.

Example 4. For any real Hilbert space H (possibly infinite-dimensional), the set $E := H\oplus\mathbb{R}$ is a Euclidean Jordan Algebra with $(a,t)*(b,s) := (sa+tb, \langle a,b\rangle + ts)$ and $\langle (a,t),(b,s)\rangle = \langle a,b\rangle + ts$. This EJA is called a *spin-factor*.

The Jordan–von Neumann-Wigner classification theorem [15] asserts that any finite-dimensional EJA is the direct sum of the finite-dimensional examples given above. This statement is still true for our possibly infinite-dimensional class of EJAs, but then the spin-factors must also be allowed to be infinite-dimensional (a proof of this fact can be found in the appendix).

Definition 5. Let E be an EJA. We call $a \in E$ *positive* (and write $a \geq 0$) when there exists a $b \in E$ such that $a = b * b$. We write $a \geq c$ when $a - c \geq 0$. We call a linear map between EJAs $f\colon E \to F$ *positive* when $f(a) \geq 0$ for all $a \geq 0$. A map f is called *unital* whenever $1_F = f(1_E)$ and *subunital* provided $f(1_E) \leq 1_F$. We let $\mathbf{EJA}_{psu}$ denote the category of Euclidean Jordan algebras with positive subunital maps between them.

For the matrix algebras the definition of positivity coincides with the regular definition of a positive matrix. For a spin factor $H \oplus \mathbb{R}$ we have $(a,t) \geq 0$ iff $t \geq \|a\|_2$, so that the set of positive elements forms the positive light-cone in a Lorentzian spacetime.

Definition 6. A positive unital linear map $\omega\colon E \to \mathbb{R}$ is called a *state* on E. An *effect* is a positive subunital linear map $a\colon \mathbb{R} \to E$ and corresponds to an $a \in E$ with $0 \leq a \leq 1$.

Because E is a Hilbert space, any map $f : E \to \mathbb{R}$ is of the form $f(a) = \langle a, b\rangle$ for some specific b. As we will see that the inner product of positive elements is positive any state on an EJA will be given by a positive $a \in E$ with $\langle a, 1\rangle = 1$. For an effect a we write $a^\perp := 1 - a$ to denote its *complement*.

We wish to define a notion of purity for the category $\mathbf{EJA}_{psu}$. The Kraus rank 1 channels from quantum theory should definitely be pure. It also seems reasonable that any isomorphism should be pure. As we are working on a Hilbert space, we have adjoints of maps. We want the pure maps to be closed under the taking of this adjoint. Finally, a composition of pure maps should remain pure, so that any composition of these examples mentioned before should be considered pure. We will see how to make these ideas exact, in particular how to generalise the idea of a Kraus rank 1 operator to arbitrary EJAs. But first we need to know more about the structure of EJAs.

Theorem 7. *Let $(E, *, 1)$ be an EJA. Sums of positive elements are again positive, so that the set of positive elements forms a* cone*. Furthermore:*

1. *The unit* 1 *is a* strong Archimedean unit*: for all $a \in E$ there exists $n \in \mathbb{N}$ such that $-n1 \leq a \leq n1$, and if $a \leq \frac{1}{n}1$ for all $n \in \mathbb{N}$ then $a \leq 0$.*
2. *As a consequence the algebra is an* order unit space *so that $\|a\| := \inf\{r \in \mathbb{R}\ ;\ -r1 \leq a \leq r1\}$ is a norm, and $a \leq \|a\|1$.*
3. *The order unit norm defined above, and the norm induced by the inner product are equivalent so that the topologies they induce are the same. The space is therefore also complete in the order unit norm topology.*
4. *The space is* self-dual*: $a \geq 0$ if and only if $\langle a, b\rangle \geq 0$ for all $b \geq 0$.*
5. *The algebra E is bounded directed complete and every state is normal so that E is in fact a* JBW-algebra *[13].*

Proof. Since to our knowledge there isn't a textbook solely dedicated to infinite-dimensional EJAs we supply a relatively self-contained proof of these claims in the appendix. □

Definition 8. For each $a \in E$ let $L_a : E \to E$ be the multiplication operator of a: $L(a)b = a * b$. We define the *quadratic representation* of the Jordan algebra by $Q_a = 2L(a)^2 - L(a^2)$.

The definition of the quadratic representation might look arbitrary, but in the case of a matrix algebra with the standard Jordan multiplication it becomes quite familiar: $Q_A B = ABA$. The quadratic representation maps will therefore act as the Jordan equivalent of the Kraus rank 1 operators of quantum theory.

Proposition 9.

1. $Q_1 = \mathrm{id}$.
2. $Q_a 1 = a^2$.
3. $\langle Q_a b, c\rangle = \langle b, Q_a c\rangle$.
4. $Q_a b = 0 \iff Q_b a = 0 \iff a * b = 0$.
5. $Q_{Q_a b} = Q_a Q_b Q_a$ *(this is known in the literature as the* fundamental equality*).*
6. *Q_a is invertible if and only a is invertible. In that case, we have $Q_a^{-1} = Q_{a^{-1}}$.*
7. *Q_a is a positive operator for all (not necessarily positive) a. If it is invertible, it is an order automorphism.*

Proof. Points 1,2 and 3 are trivial. Point 4 can be found in [1, Lemma 1.26]. Point 5 follows from MacDonalds theorem (see [13, Theorem 2.4.13]). Point 6 is proven in [1, Lemma 1.23] and finally 7 is given by [1, Theorem 1.25]. □

Note that while the Jordan multiplication maps L_a are not positive, and therefore are not maps in the category $\mathbf{EJA}_{psu}$, the quadratic representation maps Q_a are positive and when $\|a\| \leq 1$ the maps are subunital so that they do happen to lie in $\mathbf{EJA}_{psu}$. These quadratic representation maps are the Jordan algebra interpretation of the Krauss rank 1 operators from quantum theory and are therefore an important class of pure maps.

As the fundamental equality $Q_{Q_a b} = Q_a Q_b Q_a$ will be important in the proofs below, it is prudent to see why it is true, so lets consider the equation in a matrix algebra. For matrices A, B and C, the expression $Q_{Q_A B} C$ is equal to $(ABA)C(ABA)$, while the righthandside of the equation $Q_A Q_B Q_A C$ is equal to $A(B(ACA)B)A$. The equality of these expression follows by associativity of matrix multiplication.

Definition 10. We call an element p of an EJA an *idempotent* iff $p^2 := p * p = p$. An idempotent is automatically positive and below the identity. We call an idempotent *atomic* if there is no non-zero idempotent strictly smaller than it. Two idempotents p and q are *orthogonal* when $p * q = 0$ or equivalently $\langle p, q\rangle = 0$.

In a matrix algebra the idempotents are precisely the projections. The quadratic representation map Q_p is then the projection map $Q_p(a) = pap$.

Proposition 11. *Let p be an idempotent. Then Q_p is idempotent, and furthermore for $0 \leq a \leq 1$ we have $Q_p a = 0 \iff \langle a, p\rangle = 0$ and $Q_p a = a \iff a \leq p \iff p * a = a$.*

Proof. The fundamental equality (number 5 of the previous proposition) implies that when p is an idempotent: $Q_{Q_p 1} = Q_{p^2} = Q_p = Q_p Q_1 Q_p = Q_p Q_p$, so that Q_p is a positive idempotent operator, symmetric with respect to the inner product. When $Q_p a = 0$ we have $0 = \langle Q_p a, 1\rangle = \langle a, Q_p 1\rangle = \langle a, p\rangle$. Now suppose $Q_p a = a$, then because we have $a \leq 1$ we also have by positivity of Q_p, $a = Q_p a \leq Q_p 1 = p$. When $a \leq p$ then by definition there is a $r \geq 0$ such that $a + r = p$. Now $0 = \langle 0, 1\rangle = \langle p * p^\perp, 1\rangle = \langle p, p^\perp\rangle = \langle a + r, p^\perp\rangle = \langle a, p^\perp\rangle + \langle r, p^\perp\rangle$. By self-duality (see theorem 7) each of these terms is positive so that we must have $0 = \langle a, p^\perp\rangle = \langle a, Q_{p^\perp} 1\rangle = \langle Q_{p^\perp} a, 1\rangle$. Since $Q_{p^\perp} a \geq 0$ this can only be the case when $Q_{p^\perp} a = 0$ so that $p^\perp * a = 0$ from which we get $p * a = a$. □

Like in quantum theory, we have a spectral theorem for elements of a Euclidean Jordan algebra.

Proposition 12. *Let a be an element of an EJA. Then there exist real numbers λ_i and orthogonal atomic idempotents p_i such that $a = \sum_{i=1}^{n} \lambda_i p_i$ for some n.*

Proof. See the appendix. □

Proposition 13. *For an effect $a = \sum_i \lambda_i p_i$ where all $\lambda_i > 0$, we define $\lceil a \rceil = \sum_i p_i$. This is the least idempotent upper bound of a. We define $\lfloor a \rfloor = \lceil a^\perp \rceil^\perp$. This is the greatest idempotent below a.*

Proof. Let q be idempotent such that $a \leq q$. Then also $\lambda_i p_i \leq q$. By proposition 11 we then have $q * (\lambda_i p_i) = \lambda_i p_i$ so that also $q * p_i = p_i$. Again by proposition 11 we conclude that $p_i \leq q$ so that $\lceil a \rceil = \sum_i p_i \leq q$.

Now suppose q is idempotent and $q \leq a$. Then $a^\perp \leq q^\perp$ and $\lceil a^\perp \rceil \leq \lceil q^\perp \rceil = q^\perp$. Then $(q^\perp)^\perp = q \leq \lceil a^\perp \rceil^\perp = \lfloor a \rfloor$ so the floor is indeed the greatest lower bound. □

3 Filters and Corners

With the preliminaries out of the way we will start to look at additional structure that is present in the category $\mathbf{EJA}_{psu}$. The proofs in this section are heavily inspired by [22, 24] where the existence of this structure was shown for the category of von Neumann algebras.

Definition 14. Let $q \in E$ be an effect. A *corner* for q is a positive subunital map $\pi : E \to \{E|q\}$ such that $\pi(1) = \pi(q)$ and that is *initial* with this property: if $g : E \to F$ is another positive subunital map such that $g(1) = g(q)$ then there must exist a unique $\overline{g} : \{E|q\} \to F$ such that $\overline{g} \circ \pi = g$.

Note 15. The name of 'corner' is inspired by the appearance of these maps when considering matrix algebras, in which case they project unto a corner of the matrix.

Definition 16. Let $q \in E$ be an effect. A *filter* for q is a positive subunital map $\xi : E_q \to E$ such that $\xi(1) \leq q$ and that is *final* with this property: if $f : F \to E$ is another positive subunital map such that $f(1) \leq q$ then there must exist a unique $\overline{f} : F \to E_q$ such that $\xi \circ \overline{f} = f$.

Since both these types of maps satisfy a universal property, they are (for a given effect) unique up to isomorphism. In particular, given a corner $\pi : E \to \{E|q\}$ and an isomorphism $\Theta : \{E|q\} \to F$ the map $\Theta \circ \pi$ is again a corner (for q), and furthermore any corner for q is of this form. Similarly when $\xi : E_q \to E$ is a filter, and we have an isomorphism $\Theta : F \to E_q$, the map $\xi \circ \Theta$ is also a filter, and any filter for q is of this form. We will see in this section that there is a canonical choice of corner and filter for every effect.

With the concepts of corners and filters in hand we can define what we mean by a pure map.

Definition 17. We call a positive subunital map between EJAs $f : V \to W$ *pure* when there exists some corner π and some filter ξ (not neccessarily for the same effect) such that $f = \xi \circ \pi$.

Note that at the moment it is not clear that this definition is closed under composition, or even whether it is inhabited.

Proposition 18. *[1, Proposition 1.43] (Peirce-decomposition) Let p be an idempotent in an EJA E, then $E_1(p) := Q_p(E) := \{Q_p(a) \ ; \ a \in E\}$ is a sub-EJA of E consisting precisely of those elements of E for which $Q_p(a) = a$.*

Definition 19. Let E be an EJA and let q be an effect. We define $\{E|q\} := E_1(\lfloor q \rfloor) = \{E|\lfloor q \rfloor\}$ and $E_q := E_1(\lfloor q^\perp \rfloor^\perp) = E_1(\lceil q \rceil)$.

Note that for an idempotent we of course have $\{E|p\} = E_p$. With these definitions for $\{E|q\}$ and E_q we can show the existence of corners and filters, after a few brief lemmas.

Lemma 20. *Let $\omega : E \to \mathbb{R}$ be a positive map such that $\omega(p) = \omega(1)$ for some idempotent p. Then $\omega(Q_p a) = \omega(a)$ for all a.*

Proof. Given such a map ω we can define $\langle a,b\rangle_\omega := \omega(a*b)$ which is a bilinear positive semi-definite form. It then satisfies the Cauchy-Schwarz inequality: $|\langle a,b\rangle_\omega|^2 \leq \langle a,a\rangle_\omega \langle b,b\rangle_\omega$. Since $\omega(p) = \omega(1)$ we also have $\omega(p^\perp) = 0$. But then $|\omega(p^\perp * a)|^2 \leq \omega(p^\perp * p^\perp)\omega(a*a) = 0$ so that $\omega(p^\perp * a) = 0$. Then obviously $\omega(p*a) = \omega(a)$ from which we also get $\omega(p*(p*a)) = \omega(a)$. Unfolding the definition of Q_p we then get $\omega(Q_p a) = \omega(a)$. □

Corollary 21. *let $g : E \to W$ be a positive map such that $g(p) = g(1)$ for some idempotent p. Then $g(Q_p a) = g(a)$ for all a.*

Proof. Follows by the previous lemma because the states separate the maps. □

Lemma 22. *Let $g : E \to F$ be a positive map such that $g(q) = g(1)$ for some effect q, then $g(\lfloor q\rfloor) = g(1)$.*

Proof. $g(q) = g(1)$ means that $g(q^\perp) = 0$. Write $q^\perp = \sum_i \lambda_i p_i$ where $\lambda_i > 0$, then $0 = g(q^\perp) = \sum_i \lambda_i g(p_i)$. Since g is a positive map and $\lambda_i > 0$ and $p_i \geq 0$ this implies that $g(p_i) = 0$. But since $\lceil q^\perp\rceil = \sum_i p_i$ by proposition 13 $g(\lceil q^\perp\rceil) = 0$, so that $g(\lfloor q\rfloor) = g(\lceil q^\perp\rceil^\perp) = g(1)$. □

Proposition 23. *Let q be an effect. Define $\pi_q : E \to \{E|q\} = E_1(\lfloor q\rfloor)$ to be $\pi_q = r \circ Q_{\lfloor q\rfloor}$ where $r : E \to E_1(\lfloor q\rfloor)$ is the orthogonal projection map with respect to the Hilbert space structure, then π_q is a corner for q. We will refer to this map as the* standard corner *for q.*

Proof. First of all we have $\pi_q(1) = (r \circ Q_{\lfloor q\rfloor})(1) = r(\lfloor q\rfloor) = r \circ Q_{\lfloor q\rfloor}(q) = \pi_q(q)$. Now suppose $g : E \to F$ is a positive subunital map such that $g(q) = g(1)$. We must show that there is a unique $\overline{g} : \{E|q\} \to F$ such that $\overline{g} \circ \pi_q = g$.

By the previous lemma $g(\lfloor q\rfloor) = g(1)$. Define $\overline{g} : E_1(\lfloor q\rfloor) \to F$ as the restriction of g. To prove that $\overline{g} \circ \pi_q = g$, i.e. we need to show that $g(a) = g(Q_{\lfloor q\rfloor} a)$ for all a, but this follows from corollary 21. For uniqueness suppose we have a $h : E_1(\lfloor q\rfloor) \to F$ such that $h \circ \pi_q = g = \overline{g} \circ \pi_q$. Let $a \in E_1(\lfloor q\rfloor)$, then we can see a as an element of E with $\pi_q(a) = a$, so that $h(a) = h(\pi_q(a)) = \overline{g}(\pi_q(a)) = \overline{g}(a)$. □

For a positive $q = \sum_i \lambda_i p_i$ we can define a positive square root $\sqrt{q} = \sum_i \sqrt{\lambda_i} p_i$. This is the unique positive element such that $\sqrt{q} * \sqrt{q} = q$.

Proposition 24. *Let q be an effect. Define $\xi_q : E_q \to E$ to be the map $\xi_q = Q_{\sqrt{q}} \circ \iota$ where ι is the inclusion $\iota : E_q = E_1(\lceil q\rceil) \to E$, then ξ_q is a filter for q. We will refer to this map as the* standard filter *for q.*

Proof. It is trivial that $\xi_q(1) = q$ so we only need to show that it is final with respect to this property: suppose $f : F \to E$ such that $f(1) \leq q$, then there must be a unique $\overline{f} : F \to E_q$ such that $\xi_q \circ \overline{f} = f$.

We first note that if $f(1) \leq q \leq \lceil q\rceil$, then for all $0 \leq p \leq 1$ we have $f(p) \leq \lceil q\rceil$ so that $f(p) \in E_1(\lceil q\rceil)$ which means that we can restrict the codomain of f to $E_1(\lceil q\rceil) = E_q$. Writing q as $q = \sum_i \lambda_i p_i$ for some $\lambda_i > 0$, we see it has a pseudo-inverse $q^{-1} = \sum_i \lambda_i^{-1} p_i$ such that $Q_{\sqrt{q^{-1}}} q = q * q^{-1} = \lceil q\rceil$. In particular $Q_{\sqrt{q^{-1}}} f(p) \leq Q_{\sqrt{q^{-1}}} q = \lceil q\rceil$. It follows that the map $\overline{f} : W \to E_q$ given by $\overline{f}(a) = Q_{\sqrt{q^{-1}}} f(a)$ is subunital and obviously $(\xi_q \circ \overline{f})(a) = Q_{\sqrt{q}} Q_{\sqrt{q^{-1}}} f(a) = Q_{\lceil q\rceil} f(a) = f(a)$ by proposition 11.

Now for uniqueness, suppose that we have a $g : F \to E_q$ such that $\xi_q \circ g = f$. Then $Q_{\lceil q \rceil} \circ \iota \circ g = Q_{\sqrt{q^{-1}}} \circ Q_{\sqrt{q}} \circ \iota \circ g = Q_{\sqrt{q^{-1}}} \circ f$. As $Q_{\lceil q \rceil}$ acts as the identity on all elements coming from $E_1(\lceil q \rceil)$ it can be removed from the expression. By taking the corestriction of both sides to $E_1(\lceil q \rceil)$ we see that $g = Q_{\sqrt{q^{-1}}} \circ f = \overline{f}$. □

For an effect $q \in E$ we note that the space associated to the standard filter of q is $E_q = E_1(\lceil q \rceil)$ while the space of the standard corner is $\{E|q\} = E_1(\lfloor q \rfloor)$. If q is not sharp these spaces don't match so that we can't compose the filter and the corner. However, by instead taking the standard corner of $\lceil q \rceil$ instead of q the spaces do match. By considering the composition $\xi_q \circ \pi_{\lceil q \rceil}$ and expanding their definitions we see that this equals $Q_{\sqrt{q}}$. Our goal of establishing that these maps are pure is therefore successful. Note also that the standard filter and the standard corner for the unit $1 \in E$ are simply the identity. Since a filter can be composed with an isomorphism and remain a filter we see that indeed all isomorphisms are pure. Now we will look at the problem of establishing that the definition of purity given in Definition 17 is indeed closed under composition.

3.1 The polar decomposition theorem

A composition of two filters is again a filter (this is proven in the abstract setting of an effectus in [25]), and that the composition of two corners is again a corner is easy to see. Suppose we know that a composition of a filter with a corner 'in the wrong order' can be written 'in the correct order', i.e. that we can always write $\pi \circ \xi$ as $\xi' \circ \pi'$ for some different filter ξ' and corner π'. Then when we have pure maps $f = \xi_1 \circ \pi_1$ and $g = \xi_2 \circ \pi_2$ their composition is $f \circ g = \xi_1 \circ \pi_1 \circ \xi_2 \circ \pi_2$ and we can interchange π_1 and ξ_2 to get a composition of two corners with two filters, which is indeed pure. So what we need to show to establish that our definition of purity is closed under composition is that filters and corners can be interchanged. It is sufficient to prove this for the standard corner and filter. To summarise, we must show that for a given effect q and sharp p there exist effects a and b such that $\pi_p \circ \xi_q = \xi_a \circ \Phi \circ \pi_b$ where Φ is some isomorphism.

The same problem of establishing that pure maps are closed under composition in von Neumann algebras is related to the existence of *polar decompositions* of elements. By applying the polar decomposition to $\sqrt{p}\sqrt{q}$ for positive p and q we have a partial isometry u such that $\sqrt{p}q\sqrt{p} = u(\sqrt{q}p\sqrt{q})u^*$. The isomorphism Φ above is then the conjugation map $a \mapsto uau^*$ restricted to the appropriate domains. Analogously, that filters and corners can be interchanged will be a consequence of a polar decomposition theorem for Euclidean Jordan algebras that we will prove below. This result might be of interest in its own right.

For the proof we first need a new notion.

Definition 25. Let $f : E \to F$ be a positive map between EJAs. The *image* of f (if it exists) is the smallest effect q such that $f(q) = f(1)$. We will denote the image of f by $\operatorname{im} f$.

Proposition 26. *Any positive map $f : E \to F$ between Euclidean Jordan algebras has an image. This image will be an idempotent.*

Proof. Because of lemma 22 a positive map f satisfies $f(q) = f(1)$ if and only if $f(\lfloor q \rfloor) = f(1)$ so we can restrict to effects which satisfy $q = \lfloor q \rfloor$, e.g. the idempotents.

By theorem 7 EJAs are JBW-algebras (see [1]) so that the idempotents form a complete lattice. Furthermore, all states are normal, meaning they preserve infima. Because the states separate the maps, all maps are also normal. We conclude that $\operatorname{im} f = \wedge\{p\,;\, p^2 = p, f(p) = f(1)\}$ exists and that $f(\operatorname{im} f) = f(\wedge\{p\,;\, f(p) = f(1)\}) = \wedge f(p) = f(1)$. □

Definition 27. Let $\Phi : E \to E$ be a positive linear map on an EJA E. We denote its dual with respect to the inner product by Φ^*, i.e. the map defined by $\langle \Phi^*(a), b\rangle = \langle a, \Phi(b)\rangle$. We call Φ a *partial isometry* when $\Phi\Phi^*$ and $\Phi^*\Phi$ are projections.

Theorem 28. *Polar Decomposition: Let p and q be positive elements in a Euclidean Jordan algebra E. There exists a partial isometry $\Phi : E \to E$ such that $Q_q Q_p = \Phi Q_{\sqrt{Q_p q^2}}$ and $\Phi(1) = \lceil Q_q p\rceil$, $\Phi^*(1) = \lceil Q_p q\rceil$ such that $\Phi^*\Phi = Q_{\lceil Q_p q\rceil}$ and $\Phi\Phi^* = Q_{\lceil Q_q p\rceil}$.*

To see how this is related to polar decomposition note that if we plug in the unit in $Q_q Q_p$ that we will get $Q_q p^2 = Q_q Q_p 1 = \Phi Q_{\sqrt{Q_p q^2}} 1 = \Phi(Q_p q^2)$. This polar decomposition theorem should not be confused with the already established notion of polar decomposition in Jordan algebras (see for instance [11, Ch. VI]) that asserts the existence of a Jordan isomorphism between any two maximal collections of orthogonal atomic idempotents in a simple EJA. In the theory of generalised probabilistic theory, this property is also known as *strong symmetry* [2].

Proof of Theorem 28. Let $\Phi = Q_q Q_p Q_{(Q_p q^2)^{-1/2}}$ so that $\Phi^* = Q_{(Q_p q^2)^{-1/2}} Q_p Q_q$because $Q_a^* = Q_a$ for all a. Then $Q_{(Q_p q^2)^{1/2}} \Phi^* = Q_{\lceil Q_p q^2\rceil} Q_p Q_q = Q_p Q_q$. By taking adjoints we then get $Q_q Q_p = \Phi Q_{\sqrt{Q_p q^2}}$ as desired. Note that

$$\Phi^*\Phi = Q_{(Q_p q^2)^{-1/2}} Q_p Q_q Q_q Q_p Q_{(Q_p q^2)^{-1/2}} = Q_{(Q_p q^2)^{-1/2}} Q_{Q_p q^2} Q_{(Q_p q^2)^{-1/2}} = Q_{\lceil Q_p q^2\rceil}$$

by application of the fundamental equality. Since $\lceil Q_p q^2\rceil = \lceil Q_p \lceil q^2\rceil\rceil = \lceil Q_p \lceil q\rceil\rceil = \lceil Q_p q\rceil$ this can be simplified to $\Phi^*\Phi = Q_{\lceil Q_p q\rceil}$. Because $\Phi^*\Phi$ is a projection we can use [17, Proposition 6.1.1] to conclude that $\Phi\Phi^*$ must be projection as well. By a simple calculation $\Phi^*(1) = \lceil Q_p q\rceil$ so that it remains to show that $\Phi(1) = \lceil Q_q p\rceil$ and that $\Phi\Phi^*(1) = \lceil Q_q p\rceil$ since this latter condition (in combination with the knowledge that $\Phi\Phi^*$ is a projection) is sufficient to conclude that $\Phi\Phi^* = Q_{\lceil Q_q p\rceil}$.

Suppose $\Phi^*(s) = 0$ then $\langle 1, \Phi^*(s)\rangle = 0 = \langle Q_p Q_q s, (Q_p q^2)^{-1/2}\rangle$. Since $\lceil (Q_p q^2)^{-1/2}\rceil = \lceil Q_p q^2\rceil$ this gives $0 = \langle Q_p Q_q s, Q_p q^2\rangle = \langle s, Q_q Q_{p^2} Q_q 1\rangle = \langle s, Q_{Q_q p^2} 1\rangle = \langle s, Q_q p^2\rangle$. We conclude that $\Phi^*(s) = 0$ if and only if $s \perp \lceil Q_q p\rceil$ so that $\operatorname{im}\Phi^* = \lceil Q_q p\rceil$. We of course also have $0 = \langle 1, \Phi^*(s)\rangle = \langle \Phi(1), s\rangle$ so that $\lceil \Phi(1)\rceil = \operatorname{im}\Phi^* = \lceil Q_q p\rceil$. Because $\langle 1, (\Phi(1))^2\rangle = \langle \Phi(1), \Phi(1)\rangle = \langle \Phi^*\Phi(1), 1\rangle = \langle \lceil Q_p q\rceil, 1\rangle = \langle \Phi^*(1), 1\rangle = \langle 1, \Phi(1)\rangle$ we conclude that $\Phi(1) = (\Phi(1))^2$ so that $\Phi(1) = \lceil \Phi(1)\rceil = \lceil Q_q p\rceil$.

By a similar argument as above we can show that $\operatorname{im}\Phi\Phi^* = \lceil (\Phi\Phi^*)(1)\rceil$ which gives $(\Phi\Phi^*)(1) \leq \operatorname{im}\Phi\Phi^* \leq \operatorname{im}\Phi^* = \lceil Q_q p\rceil$. For the other direction we recall that we had $Q_q Q_p = \Phi Q_{\sqrt{Q_p q^2}}$ so that $Q_{Q_q p^2} = Q_q Q_p Q_p Q_q = \Phi Q_{\sqrt{Q_p q^2}} Q_{\sqrt{Q_p q^2}} \Phi^* = \Phi Q_{Q_p q^2} \Phi^* \leq \left\| Q_{Q_p q^2} \right\| \Phi\Phi^*$. By inserting the unit into the expression and taking the ceiling we are left with $\lceil Q_q p\rceil = \lceil Q_{Q_q p^2} 1\rceil \leq \lceil (\Phi\Phi^*)(1)\rceil$. □

Proposition 29. *Let $\xi_q : E_1(\lceil q\rceil) \to E$, $\xi_q = Q_{\sqrt{q}} \circ \iota$ be the standard filter of an effect q and $\pi_p : E \to E_1(p)$, $\pi_p = r \circ Q_p$ be the standard corner of a sharp effect p. Then $\pi_p \circ \xi_q = \xi_a \circ \Phi \circ \pi_b$ where a and b are some effects and Φ is an isomorphism. In other words: $\pi_p \circ \xi_q$ is pure.*

Proof. Define the shorthand $q\&p := Q_{\sqrt{q}}(p)$. Let $f = \pi_p \circ \xi_q : E_{\lceil q\rceil} \to E_p$. Because $f(1) = \pi_p(\xi_q(1)) = \pi_p(q) = p\&q$ we see that there must exist $\overline{f} : E_{\lceil q\rceil} \to E_{\lceil p\&q\rceil}$ such that $\xi_{p\&q} \circ \overline{f} = f$ where $\xi_{p\&q} : E_{\lceil p\&q\rceil} \to E_p$ by the universal property of the filter. This $\overline{f}$ is given by $\overline{f} = Q_{(p\&q)^{-1/2}} \circ f$ so that $\overline{f}(1) = Q_{(p\&q)^{-1/2}}(p\&q) = \lceil p\&q\rceil = 1$ since the codomain is $E_{\lceil p\&q\rceil}$. We will ignore the restriction and inclusion maps present in the filter and corner so that we can write $f = Q_p Q_{\sqrt{q}}$ and similarly $\overline{f} = Q_{(p\&q)^{-1/2}} Q_p Q_{\sqrt{q}}$.

Similar to the argument used in the proof of theorem 28 we can show that $\operatorname{im}\overline{f} = \lceil q\&p \rceil$. Then we can use the universal property of the corner to find a map $\Phi : E_{\lceil q\&p \rceil} \to E_{\lceil p\&q \rceil}$ such that $\Theta \circ \pi_{\lceil q\&p \rceil} = \overline{f}$. Because $\overline{f}$ and $\pi_{\lceil q\&p \rceil}$ are unital, Φ has to be unital as well. Note that Φ is just a restriction of $\overline{f}$ to the appropriate domain and that $\overline{f} = Q_{(p\&q)^{-1/2}} Q_p Q_{\sqrt{q}}$ is exactly the same as Φ^* in the proof of theorem 28. We can conclude as a consequence that $\Phi\Phi^* = Q_{\lceil p\&q \rceil}$ while $\Phi^*\Phi = Q_{\lceil q\&p \rceil}$. These are of course the identity maps on $E_{\lceil p\&q \rceil}$ respectively $E_{\lceil q\&p \rceil}$ so that $\Phi^* = \Phi^{-1}$. We conclude that $f = \xi_{p\&q} \circ \overline{f} = \xi_{p\&q} \circ \Phi \circ \pi_{\lceil q\&p \rceil}$ where Φ is an isomorphism. □

Corollary 30. *The composition of pure maps is pure.*

Proof. Let f_1 and f_2 be pure, then $f_i = \xi_i \circ \Theta_i \circ \pi_i$, so that $f_1 \circ f_2 = \xi_1 \circ \Theta_1 \circ \pi_1 \circ \xi_2 \circ \Theta_2 \circ \pi_2 = \xi_1' \circ \xi' \circ \Theta' \circ \pi' \circ \pi_2'$ by the previous proposition and writing $\xi_1 \circ \Theta_1 = \xi_1'$ and $\Theta_2 \circ \pi_2 = \pi_2'$ where ξ_1' and π_2' are again a filter respectively a corner. But now since a composition of filters is again a filter and a composition of corners is again a corner we see that $f_1 \circ f_2$ is indeed pure. □

4 Diamond adjointness and positivity

Since Euclidean Jordan algebras are also Hilbert spaces, we can find for any positive map an adjoint with respect to the inner product. This means that the category of all EJAs with positive (not necessarily subunital) maps is a dagger category. The adjoint of a subunital map is not necessarily subunital again however, so that $\mathbf{EJA}_{psu}$ is *not* a dagger category. However, the set of pure maps is closed under taking adjoints (which can be shown by a simple case analysis), so that this restricted category *is* a dagger category. When the category of positive subunital maps between von Neumann algebras is restricted to the pure maps it also forms a dagger category [24]. This is not a coincidence as the category of von Neumann algebras is a †-*effectus*. The definition of †-effectoi has been introduced as an abstract version of the category of von Neumann algebras [25]. A consequence of the results in this section will be that $\mathbf{EJA}_{psu}$ is also a †-effectus. An important notion in an effectus is that of $\diamond$-adjointness. This is a possibilistic alternative to adjointness that can be defined even when there is no obvious choice of dagger.

In this section we will study $\diamond$-adjointness in $\mathbf{EJA}_{psu}$ and show that it behaves similarly to $\diamond$-adjointness in von Neumann algebras. In particular we will give a characterisation of pure $\diamond$-self-adjoint maps and show that a pure $\diamond$-*positive* map $f : E \to E$ is completely determined by its image at the unit: $f = Q_{\sqrt{f(1)}}$ and thus that the only pure $\diamond$-positive maps are the quadratic representation maps Q_a for some positive a. As these quadratic representation maps are the Jordan equivalent of the *sequential product* map $b \mapsto aba$ [27], this can be seen as a new characterisation of the sequential product.

Definition 31. Let $f : E \to F$ be a positive subunital map and write $\operatorname{Idem}(E)$ for the set of idempotents of E. Define the maps $f^\diamond\colon \operatorname{Idem}(F) \to \operatorname{Idem}(E)$ and $f_\diamond\colon \operatorname{Idem}(E) \to \operatorname{Idem}(F)$ by

$$f^\diamond(p) \;=\; \lceil p \circ f \rceil \qquad \text{and} \qquad f_\diamond(q) \;=\; \operatorname{im}(Q_q \circ f).$$

We say that $f : E \to F$ is $\diamond$-*adjoint* to $g : F \to E$ when $f^\diamond = g_\diamond$ or equivalently $f_\diamond = g^\diamond$ [25]. We call $f : E \to E$ $\diamond$-*self-adjoint* when f is $\diamond$-adjoint to itself, and we call f $\diamond$-*positive* when there exists a $\diamond$-self-adjoint g such that $f = g \circ g$.

It can be shown that $f^\diamond(p) \leq q^\perp$ iff $f_\diamond(q) \leq p^\perp$ so that the diamond defines a Galois connection between the orthomodular lattices of idempotents. As a result we get a functor $\diamond : \mathbf{EJA}_{psu} \to \mathbf{OMLatGal}$ [25].

We start by noting that $\diamond$-self-adjointness is weaker then regular self-adjointness.

Proposition 32. *Any self-adjoint operator $f : E \to E$ on an EJA E is $\diamond$-self-adjoint. In particular Q_a is $\diamond$-self-adjoint for any $a \in E$. Consequently, Q_a is $\diamond$-positive for positive a.*

Proof. Let f be any self-adjoint operator. It suffices to show $f^\diamond(s) \leq t^\perp \iff f^\diamond(t) \leq s^\perp$ for all idempotents $s,t \in E$ (see [25, §207III]). This is equivalent to

$$\langle f^\diamond(s), t\rangle \;=\; 0 \quad \iff \quad \langle s, f^\diamond(t)\rangle \;=\; 0 \qquad (s,t \in E \text{ idempotents}). \tag{1}$$

By the spectral theorem $\langle \lceil q \rceil, s\rangle = 0 \iff \langle q,s\rangle = 0$ for any positive q and idempotent s, so (1) is equivalent to $\langle f(s),t\rangle = 0 \iff \langle s, f(t)\rangle = 0$, which clearly holds as f is self-adjoint.

Pick any positive $a \in E$. By the fundamental identity, we have $Q_a = Q_{\sqrt{a}^2} = Q_{\sqrt{a}}^2$, so Q_a is the square of a $\diamond$-self-adjoint map, hence $\diamond$-positive. □

We will now work towards characterising the pure $\diamond$-positive maps. We will start with a well-known fact about order-isomorphisms in Jordan algebras.

Lemma 33. *An effect p is called* order-sharp *when $q \leq p$ and $q \leq p^\perp$ implies that $q = 0$. An effect p is order-sharp if and only if it is an idempotent.*

Proof. Let a be an order-sharp effect and write $a = \sum_i \lambda_i p_i$. Let $r_i = \min\{\lambda_i, 1-\lambda_i\}$, then $r_i p_i \leq a$ and $r_i p_i \leq a^\perp = 1-a$ which implies that $r = 0$, so either $\lambda_i = 1$ or $\lambda_i = 0$ for all i. But then as a is a sum of orthogonal idempotents it is also an idempotent. For the other direction suppose a is sharp. Let $q \leq a$ and $q \leq a^\perp$. By $q \leq a$ we know that $Q_a q = q$, but we also have $Q_a q \leq Q_a a^\perp = 0$ so that $q = 0$. □

Proposition 34. *A unital order isomorphism between EJAs is a Jordan isomorphism: that is, it preserves the Jordan multiplication.*

Proof. Let $\Theta\colon E \to F$ be any unital order-isomorphism between EJAs. As $2(a*b) = (a+b)^2 - a^2 - b^2$ it suffices to show $\Theta(a)^2 = \Theta(a^2)$ for any $a \in E$. Write $\sum_i \lambda_i p_i = a$ for the spectral decomposition of a. As idempotents are exactly the order-sharp elements by the previous lemma and idempotents p,q are orthogonal iff $p \leq 1-q$, we see that $\Theta(p_i)$ are also pairwise orthogonal idempotents. Thus $\Theta(a)^2 = (\sum_i \lambda_i \Theta(p_i))^2 = \Theta(\sum_i \lambda_i^2 p_i) = \Theta(a^2)$, as desired. □

Corollary 35. *Let $\Theta : E \to F$ be any unital order-isomorphism between Euclidean Jordan Algebras. Then, for any $a,b \in E$, we have $\Theta(Q_a b) = Q_{\Theta(a)}\Theta(b)$. That is: $\Theta \circ Q_a = Q_{\Theta(a)} \circ \Theta$. Equivalently: $Q_a \circ \Theta = \Theta \circ Q_{\Theta^{-1}(a)}$.*

The next few results involve the notion of *faithfulness*. A map $f : E \to F$ is called faithful when for any positive a the equation $f(a) = 0$ implies $a = 0$. A map $f$is faithful if and only if $\operatorname{im} f = 1$.

Lemma 36. *Let $f : E \to E$ be a faithful pure $\diamond$-self-adjoint map between EJAs. Then $f = Q_{\sqrt{f(1)}} \circ \Theta$ for some unital Jordan isomorphism Θ with $\Theta(\sqrt{f(1)}) = \sqrt{f(1)}$ and $\Theta = \Theta^{-1}$.*

Proof. First, we collect some basic facts. As f is pure, we have $f = \xi \circ \pi$ for some filter ξ and corner π. Note that $\operatorname{im} f = \operatorname{im} \pi$ as ξ is faithful and $f(1) = \xi(1)$ as π is unital. Hence, by $\diamond$-self-adjointness of f, we have $\lceil \xi(1) \rceil = \lceil f(1) \rceil = f^\diamond(1) = f_\diamond(1) = \operatorname{im} f = \operatorname{im} \pi$. Next, by the universal properties of filters and corners, there exist order isomorphisms Θ_1, Θ_2 such that $\xi = \xi_{\xi(1)} \circ \Theta_1$ and $\pi = \Theta_2 \circ \pi_{\operatorname{im}\pi}$, so that $f = \xi_{\xi(1)} \circ \Theta_1 \circ \Theta_2 \circ \pi_{\operatorname{im}\pi} = \xi_{f(1)} \circ \Theta \circ \pi_{\operatorname{im} f}$ defining $\Theta := \Theta_1 \circ \Theta_2$.

We assumed f is faithful, ie. $\operatorname{im} f = 1$. So $\pi_{\operatorname{im} f} = \pi_1 = \mathrm{id}$. For brevity, write $q := \sqrt{f(1)}$. As $\lceil q \rceil = \lceil f(1) \rceil = \lceil \xi(1) \rceil = \operatorname{im} \pi = 1$, we have $\xi_q = Q_q$ and so $f = Q_q \circ \Theta$.

As seen in the proof of the previous proposition if f is $\diamond$-self-adjoint we have $\langle f(a),b\rangle = 0 \iff \langle a, f(b)\rangle = 0$. In this case this translates to $0 = \langle Q_q\Theta(a),b\rangle = 0 \iff 0 = \langle a, Q_q\Theta(b)\rangle = \langle Q_q a, \Theta(b)\rangle = \langle \Theta^{-1}Q_q a, b\rangle$. This implies that $\lceil Q_q\Theta(a)\rceil = \lceil \Theta^{-1}Q_q a\rceil = \Theta^{-1}(\lceil Q_q a\rceil)$ for all a. Write $q = \sum_i \lambda_i q_i$ where the q_i are atomic. Then we have $Q_q q_i = \lambda_i^2 q_i$. Filling in $a = q_i$ we then get $\lceil Q_q\Theta(q_i)\rceil = \Theta^{-1}(\lceil Q_q q_i\rceil) = \Theta^{-1}(\lceil \lambda_i^2 q_i\rceil) = \Theta^{-1}(q_i)$. The righthandside is atomic as Jordan isomorphisms preserve atomicity, so the lefthandside must also be atomic. Since $\lceil b\rceil$ is atomic if and only if b is proportional to an atomic predicate we then get $Q_q\Theta(q_i) = \mu_i\Theta^{-1}(q_i)$ for some $0 < \mu_i < 1$. Now we bring the Θ^{-1} to the lefthandside and note that:

$$\sum_i \lambda_i^2\Theta(q_i) = \Theta(q^2) = \Theta Q_q\Theta(1) = \sum_i \Theta Q_q\Theta(q_i) = \sum_i \mu_i q_i$$

Now let $p_j = \sum_{i,\lambda_i=\lambda_j} q_i$ and $r_j = \sum_{i,\mu_i=\mu_j} q_j$. Then we can write $\sum_i \lambda_i^2\Theta(q_i) = \sum_j \lambda_j^2\Theta(p_j)$ and $\sum_i \mu_i q_i = \sum_j \mu_j r_j$ where in the sums on the righthandside the λ_j and μ_j are distinct. Since Θ preserves orthogonality this means we get two orthogonal decompositions that are equal: $\sum_j \lambda_j^2\Theta(p_j) = \sum_j \mu_j r_j$. By uniqueness of such decompositions we then have $\lambda_j^2 = \mu_j$ and $\Theta(p_j) = r_j$ (where we assume for now that we have ordered the eigenvalues from high to low). But of course since the λ_j^2 and μ_j agree, the p_j and the r_j will also agree by their definition, so that $\Theta(p_j) = p_j$. But then $\Theta(q) = \sum_j \lambda_j\Theta(p_j) = \sum_i j\lambda_j p_j = q$.

Now $\Theta Q_q = Q_{\Theta(q)}\Theta = Q_q\Theta$ so the Θ commutes with Q_q. Note that since $\lceil q\rceil = 1$, q will be invertible. Let $g = Q_q$. Then $g^\diamond\Theta^\diamond = (\Theta g)^\diamond = (g\Theta)^\diamond = f^\diamond = f_\diamond = (g\Theta)_\diamond = g_\diamond\Theta_\diamond = g^\diamond(\Theta^{-1})^\diamond$. Now g^{-1} is not a subunital map, but it can be scaled downwards until it is, in which case $gg^{-1} = \lambda\mathrm{id}$ for some $\lambda > 0$, in which case $g^\diamond(g^{-1})^\diamond = \mathrm{id}$. Since $g^\diamond$ has an inverse, we see that $g^\diamond\Theta^\diamond = g^\diamond(\Theta^{-1})^\diamond$ can only hold when $\Theta^\diamond = (\Theta^{-1})^\diamond$. Since Θ is a sharp map we then have $\Theta = \Theta^{-1}$. □

Proposition 37. *Let $g : E \to E$ be a faithful pure $\diamond$-positive map with $g(1) = p$, then $g = Q_{\sqrt{p}}$.*

Proof. Since g is $\diamond$-positive there must exist some pure $\diamond$-self-adjoint $f : E \to E$ such that $g = ff$. Since g is faithful, the f must be faithful as well since $1 = \mathrm{im}\, g = \mathrm{im}\, ff \leq \mathrm{im}\, f$. By the previous lemma $f = Q_q\Theta$ for some q and $\Theta(q) = q$ and $\Theta = \Theta^{-1}$. But then $g = ff = Q_q\Theta Q_q\Theta = Q_q Q_{\Theta(q)}\Theta\Theta = Q_q Q_q\Theta^{-1}\Theta = Q_{q^2}$. Now $g(1) = Q_{q^2}1 = q^4$, so that $g = Q_{\sqrt{p}}$ where $p = q^4$. □

Theorem 38. *Let $g : E \to E$ be a pure $\diamond$-positive map with $g(1) = p$, then $g = Q_{\sqrt{p}}$.*

Proof. Let f be $\diamond$-self-adjoint so that in particular $\mathrm{im}\, f = \lceil f(1)\rceil$. We can then corestrict f to $E \to E_1(\mathrm{im}\, f)$. Using corollary 21 we also get $f = fQ_{\mathrm{im}\, f}$ so that we can factor f as $f = \xi_{\mathrm{im}\, f} \circ \overline{f} \circ \pi_{\mathrm{im}\, f}$ where $\overline{f} : E_1(\mathrm{im}\, f) \to E_1(\mathrm{im}\, f)$. Note that $\xi_{\mathrm{im}\, f}$ is nothing but the inclusion map into E. It is also easy to see that $\overline{f} = \pi_{\mathrm{im}\, f} \circ f \circ \xi_{\mathrm{im}\, f}$. Then $\overline{f}(1) = \pi_{\mathrm{im}\, f}(f(\mathrm{im}\, f)) = f(\mathrm{im}\, f) = f(1)$. Because f is pure, $\overline{f}$ is also pure as it is a composition of pure maps. When f is $\diamond$-self-adjoint we have $\overline{f}^\diamond = (\pi_{\mathrm{im}\, f} \circ f \circ \xi_{\mathrm{im}\, f})^\diamond = \xi_{\mathrm{im}\, f}^\diamond \circ f^\diamond \circ \pi_{\mathrm{im}\, f}^\diamond = (\pi_{\mathrm{im}\, f})_\diamond \circ f_\diamond \circ (\xi_{\mathrm{im}\, f})_\diamond = \overline{f}_\diamond$. Here we have used that $\xi_s^\diamond(t) = \lceil \xi_s(t)\rceil = \lceil t\rceil = t$ since $t \in E_1(s)$ and $(\pi_s)_\diamond(t) = \mathrm{im}\, \pi_t \circ \pi_s = \mathrm{im}\, \pi_t = t$ which also follows because $t \leq s$, so that $\xi_s^\diamond = (\pi_s)_\diamond$.

When f is $\diamond$-self-adjoint we get $\mathrm{im}\, f^2 = f^\diamond(f_\diamond(1)) = f^\diamond(\mathrm{im}\, f) = f^\diamond(1) = f_\diamond(1) = \mathrm{im}\, f$. For a $\diamond$-self-adjoint f we then get $\overline{f}^2 = \pi_{\mathrm{im}\, f} \circ f \circ \xi_{\mathrm{im}\, f} \circ \pi_{\mathrm{im}\, f} \circ f \circ \xi_{\mathrm{im}\, f} = \pi_{\mathrm{im}\, f} \circ f^2 \circ \xi_{\mathrm{im}\, f} = \overline{f^2}$. We conclude that when $g = f \circ f$ is $\diamond$-positive, $\overline{g} = \overline{f^2} = \overline{f}^2$ is also $\diamond$-positive. Since $\overline{g}$ is faithful we have already established that $\overline{g} = Q_{\sqrt{p}}$ where $p = \overline{g}(1) = g(1)$ and $\mathrm{im}\, g = \lceil g(1)\rceil = \lceil p\rceil$. Now $g = \xi_{\mathrm{im}\, g}Q_{\sqrt{p}}\pi_{\mathrm{im}\, g} = Q_{\sqrt{p}}Q_{\lceil p\rceil} = Q_{\sqrt{p}}$. □

5 Conclusion

We have shown that the definition of purity for von Neumann algebras in [24, 25] also works well in the category of positive subunital maps between Euclidean Jordan algebras. In particular, this definition of purity is closed under composition and the dagger, and includes the equivalents of the Krauss rank one operators for Jordan algebras. We have also shown that the possibilistic notion of $\diamond$-adjointness from effectus theory translates to these algebras, and we have used it to give a new characterisation of the 'sequential product' maps $b \mapsto Q_a b$ for positive a.

With the results in this paper we know that the definition of purity of [24, 25] works in von Neumann algebras and in Euclidean Jordan algebras. This begs the question whether our results concerning pure maps generalises to this to the class of *JBW-algebras* [13], which includes both all von Neumann algebras and EJAs. We consider this to be a challenging topic for future research.

The precise relationship between the different notions of purity found in the literature and the one studied in this paper is yet to be determined. When restricted to complex matrix algebras all the definitions coincide, but when direct sums of simple algebras are considered the definitions sometimes diverge. For example, the identity map on a direct sum is not atomic and therefore not pure in the sense of [5]. Also, the adjoint of a pure map in the sense of [9, 20] on a direct sum need not be pure.

Then finally there is the matter of the different notions of positivity for maps between Euclidean Jordan algebras: they can be *superoperator positive* (mapping positive elements to positive elements), *operator positive* as linear maps between Hilbert spaces, and, of course, $\diamond$-positive. Any relation? While preliminary investigations reveal none between superoperator positivity and operator positivity, $\diamond$-positivity does seem to be connected with operator positivity.

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A Basic structure of EJAs

EJAs are commonly defined to be finite-dimensional. The infinite-dimensional algebras we study are also known as JH-algebras [8, 7] (where we additionally require the existence of a unit). In this appendix we will give a relatively self-contained proof that the EJAs we use are JBW-algebras and that every element has a finite spectral decomposition.

Proposition 39. *For every EJA E there is a constant $r>0$ such that, for all $a,b\in E$,*

$$\|a*b\|_2 \;\leq\; r\,\|a\|_2\,\|b\|_2, \tag{2}$$

where $\|c\|_2 \equiv \sqrt{\langle c,c\rangle}$ denotes the Hilbert norm. In particular, $*$ is uniformly continuous with respect to the Hilbert norm.

Proof. The trick is to apply the *uniform boundedness principle* (see e.g. [21]) twice, which states that any collection $\mathscr{T}$ of bounded operators from a Banach space X to a normed vector space Y that is bounded pointwise, i.e. $\sup_{T\in\mathscr{T}}\|Tx\|<\infty$ for all $x\in X$, is uniformly bounded in the sense that $\sup_{T\in\mathscr{T}}\|T\|<\infty$.

For the moment fix $a\in E$. Our first step is to show that the operator $a*(\,\cdot\,)\colon E\to E$ is bounded. To this end consider the collection of linear functionals $\langle b, a*(\,\cdot\,)\rangle : E\to\mathbb{R}$, where $b\in E$ with $\|b\|_2\leq 1$. These are bounded operators, since $\langle b,a*(\,\cdot\,)\rangle = \langle a*b,(\,\cdot\,)\rangle$, and as a collection they are bounded pointwise, since $|\langle b,a*c\rangle| \leq \|b\|_2\,\|a*c\|_2 \leq \|a*c\|_2 < \infty$. Hence

$$r_a \;:=\; \sup_{\|b\|_2\leq 1} \|\,\langle a*b,(\,\cdot\,)\rangle\,\| \;<\; \infty$$

by the uniform boundedness principle. Since in particular $(\|a*b\|_2)^2 = \langle a*b,a*b\rangle \leq r_b\|a*b\|_2$ for all $b\in E$ with $\|b\|_2\leq 1$, we get $\|a*b\|_2\leq r_a$ for all $b\in E$ with $\|b\|_2\leq 1$, and thus $\|a*b\|_2\leq r_a\|b\|_2$ for any $b\in E$. In other words, the linear operator $a*(\,\cdot\,)\colon E\to E$ is bounded.

Now, to prove equation (2) it suffices to show that $\sup_{\|a\|_2\leq 1}\|a*(\,\cdot\,)\|$ is finite. For this, in turn, it suffices, by the uniform boundedness principle, to show given $b\in E$ that $\sup_{\|a\|_2\leq 1}\|a*b\|_2<\infty$. Since $\|a*b\|_2 = \|b*a\|_2 \leq \|b*(\,\cdot\,)\|\,\|a\|_2 \leq \|b*(\,\cdot\,)\| < \infty$ for all $a\in E$ with $\|a\|_2\leq 1$, this is indeed the case. □

To proceed we need some basic algebraic properties of Jordan algebras, which are most conveniently expressed with some additional notation.

Notation 40. Let E be a Jordan algebra.

1. We write $a^0 := 1,\quad a^1 := a,\quad a^2 := a*a,\quad a^3 := a*a^2,\quad a^4 := a*a^3,\ldots$. Note that since $*$ is not associative it's not clear whether equations like $a^4 = a^2*a^2$ hold.
2. Given $a\in E$ we denote the linear operator $E\to E\colon b\mapsto a*b$ by L_a.
   Given two linear operators $S,T\colon E\to E$ we write $[S,T] := ST-TS$ for the commutator of S and T.

Proposition 41. *Given a Jordan algebra E, and $a,b,c\in E$, we have*

1. $[L_a,L_{a^2}]=0;\quad [L_b,L_{a^2}] = 2[L_{a*b},L_a];\quad$ *and* $\quad [L_a,L_{b*c}]+[L_b,L_{c*a}]+[L_c,L_{a*b}]=0;$
2. $L_{a*(b*c)} = L_aL_{b*c}+L_bL_{c*a}+L_cL_{a*b}-L_bL_aL_c-L_cL_aL_b;$
3. $a^n*(b*a^m) = (a^n*b)*a^m\quad$ *and* $\quad a^n*a^m = a^{n+m}\quad$ *for all $n,m\in\mathbb{N}$.*

Proof. 1. The first equation, $[L_a,L_{a^2}]=0$, is just a reformulation of the Jordan identity:

$$L_aL_{a^2}b \;\equiv\; a*(b*a^2) \;=\; (a*b)*a^2 \;\equiv\; L_{a^2}L_ab.$$

Note that $[L_{a+b},L_{(a+b)^2}]-[L_{a-b},L_{(a-b)^2}] = 4[L_a,L_{a*b}]+2[L_b,L_{a^2}]+2[L_b,L_{b^2}]$—just expand both sides. Applying $[L_d,L_{d^2}]=0$ with $d=b,\,a+b,\,a-b$, we get $[L_b,L_{a^2}] = -2[L_a,L_{a*b}] = 2[L_{a*b},L_a]$. Similarly, one gets $[L_a,L_{b*c}]+[L_b,L_{c*a}]+[L_c,L_{a*b}]=0$ by expanding $2[L_{(a+c)*b},L_{a+c}]-[L_b,L_{(a+c)^2}]\equiv 0$.

2. Since $[L_a, L_{b*c}] + [L_b, L_{c*a}] + [L_c, L_{a*b}] = 0$, we have, for all $d \in E$,

$$(L_a L_{b*c} + L_b L_{c*a} + L_c L_{a*b})d \; = \; (b*c)*(a*d) + (c*a)*(b*d) + (a*b)*(c*d).$$

Since the right-hand side of this equation is invariant under a switch of the roles of a and d, so must be the left-hand side, which gives us the professed equality after some rewriting:

$$\begin{aligned}(L_a L_{b*c} + L_b L_{c*a} + L_c L_{a*b})d \; &= \; (L_d L_{b*c} + L_b L_{c*d} + L_c L_{d*b})a && a \leftrightarrow d \\ &\equiv \; (L_{a*(b*c)} + L_b L_a L_c + L_c L_a L_b)d && \text{rewriting.}\end{aligned}$$

3. By repeatedly applying the equation for $L_{a*(b*c)}$ from 2 it is clear that L_{a^n} and L_{a^m} may both be written as polynomial expressions in L_a and L_{a^2}. Since L_a and L_{a^2} commute by the Jordan identity, so will L_{a^n} and L_{a^m} commute. Whence $a^n*(b*a^m) = (a^n*b)*a^m$.

Finally, seeing that $a^n*a^m = a^{n+m}$ is only a matter of induction over m. Indeed, $a^n*a^0 = a^n*1 = a^n$, and if $a^n*a^m = a^{n+m}$ for all n for some fixed m, we get $a^n*a^{m+1} = a^n*(a*a^m) = (a^n*a)*a^m = a^{n+1}*a^m = a^{n+m+1}$. □

Corollary 42. *Let $a \in E$ be an element of an EJA. Let $C(a)$ denote the closure of the algebra generated by a, then $C(a)$ is a commutative associative algebra.*

Proof. Point 3 of Proposition 41 allows us to see that the smallest Jordan subalgebra of E that contains a consists of all real polynomials $\sum_{n=0}^N \lambda_n a^n$ over a, and is therefore associative. Since the Jordan multiplication is continuous (by Proposition 2) the closure $C(a)$ of this associate subalgebra will again be an associative subalgebra. □

Proposition 43. *An associative EJA is isomorphic as an algebra to $\mathbb{R}^n$ with pointwise multiplication for some $n \in \mathbb{N}$.*

Proof. Let E be an associative EJA and let $L_a : E \to E$ denote the Jordan multiplication operator of $a \in E$. This gives rise to a map $L : E \to B(E)$ that is linear (since $L_{a+b} = L_a + L_b$ and $L_{\lambda a} = \lambda L_a$), multiplicative (by associativity $L_{a*b} = L_a L_b$), unital ($L_1 = \mathrm{id}$), injective (since $L_a 1 = a$) and positive (L_a is self-adjoint and $L_{a^2} = L_a^2$ is therefore a positive operator). The map is also order-reflecting. To see this we first note that the algebra $C(L_a)$ generated by L_a in $B(E)$ is equal to the set $L(C(a)) := \{L_b \; ; \; b \in C(a)\}$. Now if $L_a \geq 0$ in $B(E)$, then it has a square root which lies in $C(L_a) = L(C(a))$, so that we can find a $b \in C(a)$ with $L_{b^2} = L_b^2 = \sqrt{L_a}^2 = L_a$ so that a is indeed positive in E. We conclude that E is order-isomorphic to some closed subspace of $B(E)$ and thus that E is a complete Archimedean order unit space.

The product of positive elements is positive, since indeed: $a^2*b^2 = (a*a)*(b*b) = (a*b)*(a*b) = (a*b)^2$. By Kadison's representation theorem [16] any complete Archimedian order unit space with unital multiplication that preserves positivity (like E), is isomorphic as an algebra to $C(X)$, the real-valued continuous functions on some compact Hausdorff space X.

Thus without loss of generality, we may assume $E = C(X)$. It is sufficient to show X is discrete (for then X must be finite by compactness). For $x \in X$, write $\delta_x \colon C(X) \to \mathbb{R}$ for the bounded linear map $\delta_x(f) = f(x)$. As $E = C(X)$ is assumed to be a Hilbert space, there must be an $\hat{x} \in C(X)$ with $\delta_x(f) = \langle \hat{x}, f \rangle = f(x)$ for all $f \in C(X)$. As $\langle \hat{x}g, f \rangle = \langle \hat{x}, gf \rangle = (gf)(x) = g(x)f(x) = \langle \hat{x}, g \rangle \langle \hat{x}, f \rangle = \langle \langle \hat{x}, g \rangle \hat{x}, f \rangle$ for all $f \in C(X)$, we must have $\hat{x}g = \langle \hat{x}, g \rangle \hat{x}$. In particular $\hat{x}\hat{y} = \langle \hat{x}, \hat{y} \rangle \hat{x}$ and with similar reasoning $\hat{x}\hat{y} = \langle \hat{x}, \hat{y} \rangle \hat{y}$. Assume $x \neq y$. Then $\hat{x} \neq \hat{y}$, but by the previous $\langle \hat{x}, \hat{y} \rangle \hat{x} = \langle \hat{x}, \hat{y} \rangle \hat{y}$. So that necessarily $0 = \langle \hat{x}, \hat{y} \rangle = \hat{x}(y)$ for all $y \neq x$. As $\hat{x} \neq 0$ and $\hat{x}$ is continuous, we see $\{x\}$ is open and so X is discrete. □

Corollary 44. *Let a be an element of an EJA. Then there exist real numbers λ_i and orthogonal idempotents p_i such that $a = \sum_{i=1}^n \lambda_i p_i$ for some n.*

Proof. Let $C(a)$ denote the EJA generated by a. This is an associative algebra by corollary 42 so that by proposition 43 we have $C(a) \cong \mathbb{R}^n$ for some n. Since $\mathbb{R}^n$ is obviously spanned by orthogonal idempotents we see that indeed $a = \sum_{i=1}^n \lambda_i p_i$. □

Proposition 45. *An element $a \in E$ is positive (i.e. a square) if and only if $\langle a,b\rangle \geq 0$ for all positive b.*

Proof. If p is an idempotent then $\langle p,a\rangle \geq 0$ if a is positive [7, p. 107]. As a result if $b = \sum_i \lambda_i p_i$ with $\lambda_i \geq 0$ and with the p_i idempotents we have $\langle a,b\rangle \geq 0$. Now for the other direction suppose $\langle a,b\rangle \geq 0$ for all positive b. Write $a = \sum_i \lambda_i p_i$ with the λ_i not necessary positive, and where the p_i are orthogonal. We then have $\langle p_i, p_j\rangle = \langle 1, p_i * p_j\rangle = \langle 1, 0\rangle = 0$ so that $0 \leq \langle a, p_j\rangle = \lambda_j \langle p_j, p_j\rangle$. Since $p_j \neq 0$ this is only possible when $\lambda_j \geq 0$. This holds for all j so that we conclude that $a \geq 0$. □

Corollary 46. *Let E be an EJA. The set of positive elements is closed under addition. More specifically E is an Archimedean order unit space.*

Proof. By the previous proposition $a \geq 0$ if and only if $\langle a,b\rangle \geq 0$ for all $b \geq 0$. But then if $c \geq 0$ we obviously have $\langle a+c, b\rangle = \langle a,b\rangle + \langle c,b\rangle \geq 0$ for all b so that indeed $a + c \geq 0$. Suppose now that $a \leq \frac{1}{n}1$ for all $n \in \mathbb{N}$. By proposition 43 the associative algebra generated by a (which contains $\frac{1}{n}1$) is isomorphic to $\mathbb{R}^n$. Since this space is Archimedian we conclude that $a \leq 0$ in $\mathbb{R}^n$ so that also $a \leq 0$ in E. In the same way we can find for any $a \in E$ a number $n \in \mathbb{N}$ so that $-n1 \leq a \leq n1$ so that E is indeed an Archimedian order unit space. □

Proposition 47. *Let E be an EJA. The topologies induced by the Hilbert norm and by the order unit norm are equivalent.*

Proof. In order to show that the topologies are the same we need to show that the norms are equivalent. Let $\|a\|$ denote the order unit norm and $\|a\|_2$ the Hilbert norm. We need to find constants $c, d \in \mathbb{R}_{>0}$ such that $c\|a\|_2 \leq \|a\| \leq d\|a\|_2$ for all $a \in E$.

Note that $\|a\|_2^2 = \langle a,a\rangle \leq \|a\|^2\langle 1,1\rangle = \|a\|^2\|1\|_2^2$ by self-duality, so that we already have one side of the inequality.

Any $a \in E$ can be written as $a = \sum_i \lambda_i p_i$ where the p_i are nonzero by the previous corollary so that $\|a\| = \max\{|\lambda_i|\}$. Now $\|a\|_2^2 = \langle a,a\rangle = \sum_i \lambda_i^2 \|p_i\|_2^2 \geq \sum_i \lambda_i^2 \inf\{\|p_j\|_2^2\} \geq \max\{|\lambda_i^2|\}\inf\{\|p_j\|_2^2\} = \|a\|^2 \inf\{\|p_j\|_2^2\}$ so if we can find some constant $R > 0$ such that $\|p\|_2 \geq R$ for all nonzero sharp effects p we are done.

Let $R = \inf_{p \neq 0, p^2 = p} \|p\|_2$. If $R \neq 0$ we are done, so suppose $R = 0$. In this case there exists a sequence of sharp effects (p_i) such that $\|p_i\|_2 \to 0$. We can then pick a subsequence such that $\|p_k\|_2 \leq 2^{-k}/k$. Now let $q_n = \sum_{i=1}^n i p_i$. Let $n \geq m$. We have $\|q_n - q_m\|_2 = \|\sum_{k=m}^n k p_k\|_2 \leq \sum_{k=m}^n k\|p_k\|_2 \leq \sum_{k=m}^n k 2^{-k}/k$ so that the (q_n) form a Cauchy sequence in the Hilbert norm. Since E is a Hilbert space it must converge to some $q \in E$ and since it is also an increasing sequence and the set of positive elements is closed in the Hilbert norm by proposition 45 we must have $q \geq q_n$ so that $\|q\| \geq \|q_n\| \geq n$ for all n which is a contradiction. □

Proposition 48. *Let E be an EJA. Then E is a JB-algebra.*

Proof. By corollary 46 E is an Archimedian order unit space. By definition E is complete in the Hilbert norm topology and by the previous proposition this topology is equivalent to the order unit topology. We conclude that E is a complete Archimedian order unit space. By [1, Theorem 1.11], E will then be a JB-algebra when the implication $-1 \leq a \leq 1 \implies 0 \leq a^2 \leq 1$ holds. So suppose $-1 \leq a \leq 1$. By the spectral theorem $a = \sum_i \lambda_i p_i$ and we must have $-1 \leq \lambda_i \leq 1$. But then $a^2 = \sum_i \lambda_i^2 p_i$ so that indeed $0 \leq a^2 \leq 1$. □

Proposition 49. *Let E be an EJA. Then E is bounded directed complete and furthermore every state is normal.*

Proof. Let $(a_i)_{i\in I}$ be a bounded upwards directed set. By translation we can take all a_i to be positive. Define for $b \geq 0$ the state $\omega(b) := \sup_{i\in I}\langle a_i, b\rangle$. This supremum exists since the a_i are bounded and the inner product between positive elements is again positive. This map can obviously be extended by linearity to the entirety of E. Since E is a Hilbert space we conclude that there must exist an $a \in E$ such that $\omega(b) = \langle a, b\rangle$ for all $b \in E$. We claim that this a is the lowest upper bound. That it is an upper bound follows by the self-duality of the order. Suppose $a_i \leq c$ for some c. Then $c - a_i \geq 0$ so that $\langle c - a_i, b\rangle \geq 0$ for all $b \geq 0$ so that $\langle c, b\rangle \geq \langle a_i, b\rangle$. By taking the supremum over the a_i's we see then that $\langle c, b\rangle \geq \langle a, b\rangle$. Again by self-duality we conclude that $c \geq a$.

For any state $\omega' : E \to \mathbb{R}$ we can find a $b \in E$ such that $\omega' = \langle \cdot, b\rangle$. As the previous argument shows, suprema of elements are defined in terms of these states so that they must preserve those suprema. □

Proposition 50. *Let p be an idempotent of an EJA. Then there exist orthogonal atomic idempotents p_i such that $p = \sum_i p_i$.*

Proof. If p is atomic we are done, so suppose it is not. Then by definition we can find $0 \leq a \leq p$ such that $a \neq \lambda p$ for some λ. Using the previous corollary write $a = \sum_i \lambda_i q_i$. If all the $q_i = p$ then $a = \lambda p$ so there must be a $q_i \neq p$. Pick this one. We have $\lambda_i q_i \leq p$. By proposition 11 we then have $Q_p(\lambda_i q_i) = \lambda_i q_i$. This of course implies $Q_p q_i = q_i$ so that again by proposition 11 we have $q_i \leq p$. We can now repeat this procedure with p replaced by q_i and $p - q_i$ to get a family of orthogonal idempotents that sum up to p. We claim that this process stops after a finite amount of iterations. By assumption the resulting idempotents are then orthogonal.

Suppose the process does not halt after a finite amount of iterations. Then we are left with a countable collection of orthogonal idempotents $(q_i)_i$. By 1 we have in any Jordan algebra for any a and b: $[L_a, L_{b^2}] + 2[L_b, L_{a*b}] = 0$ Let $a = q_i$ and $b = q_j$ with $i \neq j$ so that $a * b = 0$, then we conclude that $[L_{q_i}, L_{q_j}] = 0$ and thus that $q_i * (a * q_j) = (q_i * a) * q_j$. The algebra spanned by the $(q_i)_i$ is therefore associative. As this algebra is necessarily infinite-dimensional this is in contradiction to proposition 43. □

Proposition 51. *Let E be an EJA. Then E is a type I JBW-algebra of finite rank.*

Proof. By proposition 48 E is a JB-algebra. By definition, a JBW-algebra is a JB-algebra that is bounded directed complete and that is separated by normal states. Proposition 49 therefore has established that E is indeed a JBW-algebra. A JBW-algebra is of type I if below every idempotent we can find an atomic idempotent. This is true by proposition 50. By this same proposition we can write the identity as a finite sum of atomic idempotents, so that the space is indeed of finite rank. □

Corollary 52. *Let E be an EJA. Then there exists a finite-dimensional EJA E_{fin} and an EJA E_{inf} that is a direct sum of infinite-dimensional spin-factors such that E is isomorphic as an EJA to $E_{fin} \oplus E_{inf}$.*

Proof. By the previous corollary E is a type I JBW-algebra of finite rank. It is therefore isomorphic to a finite direct sum of type I JBW-factors of finite rank. These factors have been classified in [13]. They are either finite-dimensional or they are infinite-dimensional spin-factors. □