Orthogonality for quantum Latin isometry squares

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Goyeneche et al recently proposed a notion of orthogonality for quantum Latin squares, and showed that orthogonal quantum Latin squares yield quantum codes. We give a simplified characterization of orthogonality for quantum Latin squares, which we show is equivalent to the existing notion. We use this simplified characterization to give an upper bound for the number of mutually orthogonal quantum Latin squares of a given size, and to give the first examples of orthogonal quantum Latin squares that do not arise from ordinary Latin squares. We then discuss quantum Latin isometry squares, generalizations of quantum Latin squares recently introduced by Benoist and Nechita, and define a new orthogonality property for these objects, showing that it also allows the construction of quantum codes. We give a new characterization of unitary error bases using these structures.

1 Introduction

1.1 Summary

At QPL 2016 the present authors introduced *quantum Latin squares* [11, 14], as quantum structures generalizing the well-known Latin squares from classical combinatorics [6]. Since then this work has been built on separately by a number of researchers: in particular, by Goyeneche, Raissi, Di Martino and Życzkowski [7], who propose a notion of *orthogonality* for quantum Latin squares which allows the construction of quantum codes; and also by Benoist and Nechita [4], who introduce *matrices of partial isometries of type (1,2,3,4)*, generalizations of quantum Latin squares which characterize system-environment observables preserving a certain set of pointer states.

In this paper we give a new formulation of orthogonality for quantum Latin squares, and use it to relate and generalize the works just cited, and extend them in certain ways. In particular, we highlight the following key contributions.

- We give a new, simplified definition of orthogonality for quantum Latin squares, and show that it is equivalent to the existing definition of Goyeneche et al [7, Definition 3]. (Definition 3, Theorem 9.)
- We give the first example of a pair of orthogonal quantum Latin squares which are not equivalent to a pair of classical Latin squares. (Example 4 and Proposition 13.)
- We show that there can be at most $n-1$ mutually orthogonal quantum Latin squares of dimension n (Theorem 18.)
- We define a notion of *quantum Latin isometry square* based on the *matrices of partial isometries of type (1,2,3,4)* defined by Benoist and Nechita [4, Definition 3.2], and define a new notion of orthogonality for these objects. (Definitions 19 and 25.)
- We show how orthogonal quantum Latin isometry squares can be used to build quantum codes. (Theorem 31.)

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- We show that unitary error bases give rise to orthogonal pairs quantum Latin isometry squares, and in fact can be characterized in terms of them. (Theorem 34.)

1.2 Related work

Since the introduction of quantum Latin squares by the present authors [14], two notions of orthogonality for quantum Latin squares have been introduced, both of which extend the standard notion for classical Latin squares.

The first such notion, to which we refer here as *left orthogonality*, was introduced by the first author [11], who showed it could be used to construct maximally entangled mutually unbiased bases. Given a pair of classical Latin squares which are left orthogonal by this definition, the left conjugates of each square are orthogonal Latin squares in the traditional sense [10]. This notion of orthogonality between QLS is not comparable to that which we study in this paper.

More recently, Goyeneche et al [7] introduced another notion of orthogonality for quantum Latin squares, which also extends the traditional definition for classical Latin squares, and the definition which we study here is equivalent. They extended their notion to quantum orthogonal arrays, more general objects which we do not consider here.

1.3 Outline

This paper has the following structure. In Section 2, we give background on quantum Latin squares, introduce our new definition of orthogonality, and explore its consequences, especially in relation to the work of Goyeneche et al [7]. In Section 3, we define quantum Latin isometry squares based on the work of Benoist and Nechita [4], and investigate a new notion of orthogonality for these objects.

2 Quantum Latin squares and orthogonality

In this section we prove our main results concerning orthogonal quantum Latin squares. In Section 2.1 we recall the definition of quantum Latin squares, give our new definition of orthogonality, and give a nontrivial example. In Section 2.2 we show that our notion of orthogonality is equivalent to a previous, more complicated definition due to Goyeneche et al [7]. In Section 2.3 we explore the connection between equivalence and orthogonality of quantum Latin squares, and show that our example of orthogonal quantum Latin squares is not equivalent to a pair of orthogonal classical Latin squares. In Section 2.4, we give a simpler definition of orthogonality for families of quantum Latin squares, and show it agrees with that due to Goyeneche et al. In Section 2.5, we prove an upper bound on the number of mutually orthogonal quantum Latin squares that can exist in any dimension.

2.1 First definitions

We begin with the definition of a quantum Latin square, recently proposed by the present authors [14].

Definition 1. A *quantum Latin square (QLS)* Ψ of dimension n is an n-by-n array of elements $|\Psi_{ij}\rangle \in \mathbb{C}^n$, such that every row and every column gives an orthonormal basis for $\mathbb{C}^n$.

A point of notation: when we write $|\Psi_{ij}\rangle$, the indices i and j refer to the row and columns of the array respectively, and take values in the set $[n] = \{0, 1, \ldots, n-1\}$.

Example 2. Here is a quantum Latin square of dimension 4, given in terms of the computational basis elements $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\} \subset \mathbb{C}^4$:

$\|0\rangle$	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$
$\frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$	$\frac{1}{\sqrt{5}}(i\|0\rangle + 2\|3\rangle)$	$\frac{1}{\sqrt{5}}(2\|0\rangle + i\|3\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle + \|2\rangle)$
$\frac{1}{\sqrt{2}}(\|1\rangle + \|2\rangle)$	$\frac{1}{\sqrt{5}}(2\|0\rangle + i\|3\rangle)$	$\frac{1}{\sqrt{5}}(i\|0\rangle + 2\|3\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$
$\|3\rangle$	$\|2\rangle$	$\|1\rangle$	$\|0\rangle$

It can readily be checked that along each row, and along each column, the elements form an orthonormal basis for $\mathbb{C}^4$. A *classical Latin square* is a quantum Latin square for which every element of the array is in the computational basis. It is easy to see that classical Latin squares are exactly the ordinary Latin squares studied in combinatorics [6], and so the theory of quantum Latin squares extends this classical theory.

There is a standard notion of orthogonality for classical Latin squares [10]. The focus of this paper is the extension of this property to quantum Latin squares, by way of the following new definition.

Definition 3. Two quantum Latin squares Φ, Ψ of dimension n are *orthogonal* just when the set of vectors $\{|\Phi_{ij}\rangle \otimes |\Psi_{ij}\rangle | i, j \in [n]\}$ form an orthonormal basis of the space $\mathbb{C}^n \otimes \mathbb{C}^n$.

We show in Theorem 9 that this agrees with a more complicated definition recently proposed by Goyeneche et al [7], in terms of partial traces of a tensor expression.

In that paper, it was shown that for classical Latin squares, this agrees with the classical notion of orthogonality. However, no non-classical examples were given of orthogonal quantum Latin squares. We now rectify this.

Example 4 (Non-classical orthogonal quantum Latin squares)**.** Define the unitary matrix U as follows:

$$U := \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & e^{\frac{2\pi i}{3}} & e^{\frac{-2\pi i}{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & e^{\frac{-2\pi i}{3}} & e^{\frac{2\pi i}{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{2} & -i\sqrt{6/7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/\sqrt{2} & -i\sqrt{3/14} & 0 & 0 & 0 \\ 0 & 0 & 0 & i & i/\sqrt{2} & \sqrt{3/14} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3/2} & \sqrt{3/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3/2} & -\sqrt{3/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} \end{pmatrix}$$

Then the following arrays are a pair of orthogonal quantum Latin squares of dimension 9:

$\|0\rangle$	$\|2\rangle$	$\|1\rangle$	$\|3\rangle$	$\|5\rangle$	$\|4\rangle$	$\|6\rangle$	$\|8\rangle$	$\|7\rangle$
$\|2\rangle$	$\|1\rangle$	$\|0\rangle$	$\|5\rangle$	$\|4\rangle$	$\|3\rangle$	$\|8\rangle$	$\|7\rangle$	$\|6\rangle$
$\|1\rangle$	$\|0\rangle$	$\|2\rangle$	$\|4\rangle$	$\|3\rangle$	$\|5\rangle$	$\|7\rangle$	$\|6\rangle$	$\|8\rangle$
$\|6\rangle$	$\|8\rangle$	$\|7\rangle$	$\|0\rangle$	$\|2\rangle$	$\|1\rangle$	$\|3\rangle$	$\|5\rangle$	$\|4\rangle$
$\|8\rangle$	$\|7\rangle$	$\|6\rangle$	$\|2\rangle$	$\|1\rangle$	$\|0\rangle$	$\|5\rangle$	$\|4\rangle$	$\|3\rangle$
$\|7\rangle$	$\|6\rangle$	$\|8\rangle$	$\|1\rangle$	$\|0\rangle$	$\|2\rangle$	$\|4\rangle$	$\|3\rangle$	$\|5\rangle$
$U\|3\rangle$	$U\|5\rangle$	$U\|4\rangle$	$\|6\rangle$	$\|8\rangle$	$\|7\rangle$	$U\|0\rangle$	$U\|2\rangle$	$U\|1\rangle$
$U\|5\rangle$	$U\|4\rangle$	$U\|3\rangle$	$\|8\rangle$	$\|7\rangle$	$\|6\rangle$	$U\|2\rangle$	$U\|1\rangle$	$U\|0\rangle$
$U\|4\rangle$	$U\|3\rangle$	$U\|5\rangle$	$\|7\rangle$	$\|6\rangle$	$\|8\rangle$	$U\|1\rangle$	$U\|0\rangle$	$U\|2\rangle$

$\|0\rangle$	$\|2\rangle$	$\|1\rangle$	$\|3\rangle$	$\|5\rangle$	$\|4\rangle$	$\|6\rangle$	$\|8\rangle$	$\|7\rangle$
$\|1\rangle$	$\|0\rangle$	$\|2\rangle$	$\|4\rangle$	$\|3\rangle$	$\|5\rangle$	$\|7\rangle$	$\|6\rangle$	$\|8\rangle$
$\|2\rangle$	$\|1\rangle$	$\|0\rangle$	$\|5\rangle$	$\|4\rangle$	$\|3\rangle$	$\|8\rangle$	$\|7\rangle$	$\|6\rangle$
$U\|6\rangle$	$U\|8\rangle$	$U\|7\rangle$	$\|0\rangle$	$\|2\rangle$	$\|1\rangle$	$\|3\rangle$	$\|5\rangle$	$\|4\rangle$
$U\|7\rangle$	$U\|6\rangle$	$U\|8\rangle$	$\|1\rangle$	$\|0\rangle$	$\|2\rangle$	$\|4\rangle$	$\|3\rangle$	$\|5\rangle$
$U\|8\rangle$	$U\|7\rangle$	$U\|6\rangle$	$\|2\rangle$	$\|1\rangle$	$\|0\rangle$	$\|5\rangle$	$\|4\rangle$	$\|3\rangle$
$\|3\rangle$	$\|5\rangle$	$\|4\rangle$	$\|6\rangle$	$\|8\rangle$	$\|7\rangle$	$U\|0\rangle$	$U\|2\rangle$	$U\|1\rangle$
$\|4\rangle$	$\|3\rangle$	$\|5\rangle$	$\|7\rangle$	$\|6\rangle$	$\|8\rangle$	$U\|1\rangle$	$U\|0\rangle$	$U\|2\rangle$
$\|5\rangle$	$\|4\rangle$	$\|3\rangle$	$\|8\rangle$	$\|7\rangle$	$\|6\rangle$	$U\|2\rangle$	$U\|1\rangle$	$U\|0\rangle$

(1)

We now consider some equivalent characterizations of orthogonality, which will be useful later.

Lemma 5. *Two quantum Latin squares Φ, Ψ are orthogonal if and only if one, and hence both, of the following equivalent conditions hold:*

$$\sum_{i,j=0}^{n-1} |\Phi_{ij}\rangle\langle\Phi_{ij}| \otimes |\Psi_{ij}\rangle\langle\Psi_{ij}| = \mathbb{I}_{n^2} \tag{2}$$

$$\sum_{i,j,p,q=0}^{n-1} \langle\Phi_{ij}|\Phi_{pq}\rangle\langle\Psi_{ij}|\Psi_{pq}\rangle|ij\rangle\langle pq| = \mathbb{I}_{n^2} \tag{3}$$

Proof. For the first condition, equation (2) says that if we sum up outer products of each element of the family $\{|\Phi\rangle_{ij} \otimes |\Psi_{ij}\rangle | i,j \in [n]\}$, we get the identity; clearly this is equivalent to the statement that the family yields an orthonormal basis. For the second condition, consider the linear map $S = \sum_{i,j} |i\rangle|j\rangle\langle\Phi_{ij}|\langle\Psi_{ij}|$, an operator on $\mathbb{C}^n \otimes \mathbb{C}^n$. The quantum Latin squares Φ,Ψ are orthogonal if and only if this map is unitary, since it transports the orthonormal basis $\{|\Phi\rangle_{ij} \otimes |\Psi_{ij}\rangle | i,j \in [n]\}$ to the computational basis. Since it is an operator on a finite-dimensional Hilbert space, S is unitary if and only if it is an isometry, and equation (3) is the isometry condition. □

Orthogonality of quantum Latin squares is unaffected by conjugation of one of the squares.

Definition 6. Given a quantum Latin square Ψ, its *conjugate* Ψ^* is the quantum Latin square with entries $(\Psi^*)_{ij} = (\Psi_{ij})^*$.

Lemma 7. *Two quantum Latin squares Φ,Ψ are orthogonal just when Φ^*,Ψ are orthogonal.*

Proof. Suppose Φ,Ψ are orthogonal quantum Latin squares. Then by equation (3), it follows that $\sum_{i,j,p,q=0}^{n-1}\langle\Phi_{ij}|\Phi_{pq}\rangle\langle\Psi_{ij}|\Psi_{pq}\rangle = \delta_{ip}\delta_{jq}$. So for all $(i,j) \neq (p,q)$ either $\langle\Phi_{ij}|\Phi_{pq}\rangle = 0$ or $\langle\Psi_{ij}|\Psi_{pq}\rangle = 0$, and we know that $\langle\Phi_{ij}|\Phi_{ij}\rangle = \langle\Psi_{ij}|\Psi_{ij}\rangle = 1$. Since $0,1 \in \mathbb{R}$, we conclude that $\sum_{i,j,p,q=0}^{n-1}\langle\Phi^*_{ij}|\Phi^*_{pq}\rangle\langle\Psi_{ij}|\Psi_{pq}\rangle = \delta_{ip}\delta_{jq}$, and hence by equation (2) it follows that Φ^*,Ψ are orthogonal. The converse then follows since $(\Phi^*)^* = \Phi$. □

2.2 Relationship to previous notion of orthogonality

The following definition of orthogonality for quantum Latin squares has recently been proposed. It is less conceptual than our Definition 3, and more complex to work with.

Definition 8 ([7], Definition 3)**.** Two quantum Latin squares Φ,Ψ are *GRMZ-orthogonal* (in their terminology, "disentangled orthogonal") when for each tensor factor $X \in \{A,B,C\}$, the following holds:

$$\mathrm{Tr}_X\left(\sum_{i,p,j=0}^{n-1} |\Phi_{ij}\rangle\langle\Phi_{ij}|_A \otimes |\Psi_{pj}\rangle\langle\Psi_{pj}|_B \otimes |i\rangle\langle p|_C\right) = \mathbb{I}_{n^2} \tag{4}$$

We now show this is equivalent to our Definition 3.

Theorem 9. *Two quantum Latin squares Φ,Ψ are orthogonal if and only if they are GRMZ-orthogonal.*

Proof. We first consider equation (4) for the case $X = C$, which yields the following equation:

$$\sum_{i,j,p=0}^{n-1} |\Phi_{ij}\rangle\langle\Phi_{pj}| \otimes |\Psi_{ij}\rangle\langle\Psi_{pj}|\langle p|i\rangle = \sum_{i,j=0}^{n-1} |\Phi_{ij}\rangle\langle\Phi_{ij}| \otimes |\Psi_{ij}\rangle\langle\Psi_{ij}|\langle ij|ij\rangle = \mathbb{I}_{n^2} \tag{5}$$

This corresponds to our equation (2). By Lemma 5, it will hold if and only if Φ,Ψ are orthogonal.

We now show that the trace conditions over $X = A$ and $X = B$ in the GRMZ-orthogonality definition are redundant, in the sense that they hold automatically for all pairs of quantum Latin squares Φ,Ψ, regardless of orthogonality. We analyze the case that $X = B$; the case $X = A$ is similar. The trace condition yields the following equation:

$$\sum_{i,j,p=0}^{n-1} \langle\Phi_{ij}|\Phi_{pj}\rangle |\Psi_{ij}\rangle\langle\Psi_{pj}| \otimes |i\rangle\langle p| = \mathbb{I}_{n^2}$$

But this equation is holds for any quantum Latin squares Φ,Ψ, as follows:

$$\begin{aligned}\sum_{i,j,p=0}^{n-1} \langle\Phi_{ij}|\Phi_{pj}\rangle |\Psi_{ij}\rangle\langle\Psi_{pj}| \otimes |i\rangle\langle p| &= \sum_{i,j,p=0}^{n-1} \delta_{ip} |\Psi_{ij}\rangle\langle\Psi_{pj}| \otimes |i\rangle\langle p| = \sum_{i,j=0}^{n-1} |\Psi_{ij}\rangle\langle\Psi_{ij}| \otimes |i\rangle\langle i| \\ = \sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-1} |\Psi_{ij}\rangle\langle\Psi_{ij}|\right) \otimes |i\rangle\langle i| &= \sum_{i=0}^{n-1} \mathbb{I}_n \otimes |i\rangle\langle i| = \mathbb{I}_n \otimes \left(\sum_{i=0}^{n-1} |i\rangle\langle i|\right) = \mathbb{I}_n \otimes \mathbb{I}_n\end{aligned}$$

Here the first equality uses the fact that Φ is a QLS, the second equality uses the definition of δ_{ij}, and the third equality rearranges the sum, the fourth equality uses the fact that Ψ is a QLS, and the final equalities are trivial algebraic manipulations. This completes the proof. □

Remark 10. Goyeneche et al also discuss the notion of "entangled orthogonal quantum Latin squares". They prove that such a thing is exactly a single 2-uniform tensor. We choose to avoid this "entangled orthogonal" terminology, since the 2-uniform terminology is well-known.

2.3 Equivalence and orthogonality

Two classical Latin squares are said to be equivalent if one can be transformed into the other by permutations of the rows, columns or computational basis state labels. Similarly, there is a notion of equivalence between quantum Latin squares [14], which we now recall.

Definition 11. Two quantum Latin squares Φ,Ψ of dimension n are *equivalent* if there exists some unitary operator U on $\mathbb{C}^n$, family of modulus-1 complex numbers c_{ij}, and permutations $\sigma,\tau \in S_n$, such that the following holds for all $i, j \in [n]$:

$$c_{ij} U |\Phi_{\sigma(i),\tau(j)}\rangle = |\Psi_{ij}\rangle \tag{6}$$

If the permutation data is the same, then orthogonality is preserved by arbitrary equivalences on each factor.

Lemma 12. *Given quantum Latin squares Φ,Ψ,Φ',Ψ' of dimension n, unitary operators U,V on $\mathbb{C}^n$, families of modulus-1 complex numbers c_{ij}, d_{ij}, and permutations $\sigma,\tau \in S_n$ such that*

$$|\Phi'_{ij}\rangle := c_{ij} U |\Phi_{\sigma(i),\tau(j)}\rangle \qquad\qquad |\Psi'_{ij}\rangle := d_{ij} V |\Psi_{\sigma(i),\tau(j)}\rangle \tag{7}$$

then Φ,Ψ are orthogonal if and only if Φ',Ψ' are orthogonal.

Proof. By Definition 3 we have the following for all i, j, m, n:

$$\begin{aligned}
&\langle \Phi_{mn}|\Phi_{ij}\rangle\langle \Psi_{mn}|\Psi_{ij}\rangle = \delta_{im}\delta_{jn} \\
&\Leftrightarrow \langle \Phi_{m'n'}|U^\dagger \circ U|\Phi_{i'j'}\rangle\langle \Psi_{m'n'}|V^\dagger \circ V|\Psi_{i'j'}\rangle = \delta_{i'm'}\delta_{j'n'}c_{m'n'}c^*_{i'j'}d_{m'n'}d^*_{i'j'} \\
&\Leftrightarrow \langle \Phi_{m'n'}|U^\dagger c^*_{m'n'} \circ c_{i'j'}U|\Phi_{i'j'}\rangle\langle \Psi_{m'n'}|V^\dagger d^*_{m'n'} \circ d_{i'j'}V|\Psi_{i'j'}\rangle = \delta_{i'm'}\delta_{j'n'} \\
&\Leftrightarrow \langle \Phi'_{mn}|\Phi'_{ij}\rangle\langle \Psi'_{mn}|\Psi'_{ij}\rangle = \delta_{im}\delta_{jn}
\end{aligned}$$

Where $\sigma(i) = i', \tau(j) = j', \sigma(m) = m'$ and $\tau(n) = n'$. □

We now show that the pair of orthogonal quantum Latin squares illustrated in Example 4 are not equivalent to any pair of orthogonal Latin squares.

Proposition 13. *The orthogonal quantum Latin squares of Example 4 are not equivalent to a pair of orthogonal classical Latin squares.*

Proof. It is enough to show that the left-hand quantum Latin square of Example 4, which we call Φ, is not equivalent to a classical Latin square. Clearly no permutation of the rows or columns could transform Φ into a classical Latin square. Suppose for a contradiction that there exists a unitary operator V and a set of phases c_{ij} such that $|\eta_{ij}\rangle := c_{ij}V|\Phi_{ij}\rangle$ are all computational basis elements, and therefore yield a classical Latin square. Then for all i, j, m, n, we must have $\langle \eta_{mn}|\eta_{ij}\rangle = 0$ or 1. We choose $m = 0$, $n = 3$, $i = 6$ and $j = 2$ to obtain $\langle \eta_{03}|\eta_{62}\rangle = c^*_{03}c_{62}\langle \Phi_{03}|V^\dagger V|\Phi_{62}\rangle = c^*_{03}c_{62}\langle \Phi_{03}|\Phi_{62}\rangle = c^*_{03}c_{62}\langle 3|U|4\rangle = c^*_{03}c_{62}\sqrt{2/3}$. But since the c_{ij} have modulus 1, this can never equal 0 or 1, and the contradiction is established. □

2.4 Generalization to multiple systems

We now extend this definition to sets of quantum Latin squares, generalising mutually orthogonal Latin squares. In particular, we show that no essentially new concept is introduced, with the existing pairwise orthogonality property being sufficient.

Definition 14 (MOQLS)**.** A family of m quantum Latin squares $\{\Phi^k | k \in [m]\}$ are *mutually orthogonal* if they are pairwise orthogonal.

Definition 15 (GRMZ-MOQLS)**.** A family of m quantum Latin squares $\{\Phi^k | k \in [m]\}$ are *GRMZ–mutually orthogonal* when the following equations hold, where X indicates a partial trace over any of the $m+1$ subsystems:

$$\mathrm{Tr}_X\left(\sum_{i,j,p,q=0}^{n-1} |\Phi^0_{ij}\rangle\langle\Phi^0_{pq}| \otimes |\Phi^1_{ij}\rangle\langle\Phi^1_{pq}| \otimes \ldots \otimes |\Phi^{m-1}_{ij}\rangle\langle\Phi^{m-1}_{pq}| \otimes |ij\rangle\langle pq|\right) = \mathbb{I}_{n^2} \tag{8}$$

Proposition 16 (MOQLS = GRMZ-MOQLS)**.** *A family of m quantum Latin squares $\{\Phi^k | k \in [m]\}$ are mutually orthogonal just when they are GRMZ–mutually orthogonal.*

Proof. We label the kth QLS system by A_k and the two other systems in equation (8) as α and β. So X can range over m element subsets of $\{A_0, A_1, \ldots, A_{m-1}, \alpha, \beta\}$. We will label such sets by the two elements that are NOT included so for example $(A_g, A_h) = \{A_0, \ldots, A_{g-1}, A_{g+1}, \ldots, A_{h-1}, A_{h+1}, \ldots A_{m-1}, \alpha, \beta\}$.

First we show that for all g and h, substituting $X=(A_g,A_h)$ into equation (8) reduces to equation (2) and so by varying g and h we obtain Definition 14. Let $X=(A_g,A_h)$ then we have:

$$\sum_{i,j,p,q=0}^{n-1}\langle\Phi^0_{pq}|\Phi^0_{ij}\rangle\cdots\langle\Phi^{g-1}_{pq}|\Phi^{g-1}_{ij}\rangle|\Phi^g_{pq}\rangle\langle\Phi^g_{ij}|\langle\Phi^{g+1}_{pq}|\Phi^{g+1}_{ij}\rangle\cdots\langle\Phi^{g-1}_{pq}|\Phi^{h-1}_{ij}\rangle|\Phi^h_{pq}\rangle\langle\Phi^h_{ij}|\langle\Phi^{h+1}_{pq}|\Phi^{h+1}_{ij}\rangle$$
$$\cdots\langle\Phi^{m-1}_{pq}|\Phi^{m-1}_{ij}\rangle\langle pq|ij\rangle=\mathbb{I}_{n^2}\qquad\Leftrightarrow\qquad\sum_{i,j=0}^{n-1}|\Phi_{ij}\rangle\langle\Phi_{ij}|\otimes|\Psi_{ij}\rangle\langle\Psi_{ij}|=\mathbb{I}_{n^2}$$

Thus equation (8) implies Definition (14) and for $X=(A_g,A_h)$ Definition 14 implies equation (8) for all g and h. We now show that Definition 14 implies equation (8) for all possible values of X.

Since $\sum_{j=0}^{n-1}\langle\Phi_{ij}|\Phi_{pj}\rangle=\delta_{ip}$ by the quantum Latin square property, we have that for all k, substituting $X=(A_k,\alpha)$ into equation (8) reduces to $\sum_{i,j=0}^{n-1}|\Phi^k_{ij}\rangle\langle\Phi^k_{ij}|\otimes|j\rangle\langle j|=\mathbb{I}_{n^2}$, which holds for all QLS. Similarly by setting $X=(A_k,\beta)$ we obtain $\sum_{i,j=0}^{n-1}|\Phi^k_{ij}\rangle\langle\Phi^k_{ij}|\otimes|j\rangle\langle j|=\mathbb{I}_{n^2}$ which again holds for all QLS. Finally we are left with $X=(\alpha,\beta)$, which gives the following:

$$\sum_{i,j,p,q=0}^{n-1}\langle\Phi^0_{ij}|\Phi^0_{pq}\rangle\ldots\langle\Phi^{m-1}_{ij}|\Phi^{m-1}_{pq}\rangle|ij\rangle\langle pq|=\mathbb{I}_{n^2}\tag{9}$$

Split the m QLS into pairs. Equation (3) is equivalent to $\sum_{i,j,p,q=0}^{n-1}\langle\Phi_{ij}|\Phi_{pq}\rangle\langle\Psi_{ij}|\Psi_{pq}\rangle=\delta_{ip}\delta_{qj}$. If m is even then the LHS of equation (9) becomes $\sum_{i,j,p,q=0}^{n-1}\delta_{ip}\delta_{pq}|ij\rangle\langle pq|$ which is a resolution of the identity. For m odd we have $\sum_{i,j}^{n-1}\langle\Phi^{m-1}_{ij}|\Phi^{m-1}_{ij}\rangle|ij\rangle\langle ij|$ which again is a resolution of the identity since all entries of a QLS are unit vectors. □

We now state the following corollary of Lemma 12 in the context of MOQLS.

Corollary 17. *Given a set of MOQLS Φ^k, the set of quantum Latin squares with entries $c_{ij}U_k|\Phi^k_{\sigma(i),\tau(j)}\rangle$ are also mutually orthogonal, for any set of unitary operators U_k, complex phases c_{ij} and permutations σ,τ.*

2.5 Upper bounds on the number of mutually orthogonal quantum Latin squares

We now show that the upper bound for the number of MOQLS of a given size is equal to the upper bound for MOLS.

Theorem 18. *Any family of MOQLS of dimension n has size at most $n-1$.*

Proof. Suppose that we have a set of m-MOQLS $|\Phi^0_{ij}\rangle,\ldots,|\Phi^{m-1}_{ij}\rangle$ of size $n\times n$. By Corollary 17 we can apply unitaries to each QLS such that the first row of every QLS is the computational basis $|i\rangle, i\in\{0,\ldots,n-1\}$. Consider $\langle\Phi^k_{10}|\Phi^l_{10}\rangle$ for some $k,l\in\{0,\ldots,m-1\}$ such that $k\neq l$. We have that:

$$\langle\Phi^k_{10}|\Phi^l_{10}\rangle=\sum_{i=0}^{n-1}\langle\Phi^k_{10}|i\rangle\langle i|\Phi^l_{10}\rangle=\sum_{i=0}^{n-1}\langle\Phi^k_{10}|\Phi^k_{0i}\rangle\langle\Phi^l_{0i}|\Phi^l_{10}\rangle$$
$$=\sum_{i=0}^{n-1}\langle\Phi^k_{10}|\Phi^k_{0i}\rangle\langle\Phi^{l*}_{10}|\Phi^{l*}_{0i}\rangle=\sum_{i=0}^{n-1}\langle\Phi^k_{10}|\Phi^k_{0i}\rangle\langle\Phi^l_{10}|\Phi^l_{0i}\rangle=\delta_{i0}\delta_{01}=0$$

So the m unit vectors $|\Phi^i_{10}\rangle$ together with $|0\rangle$ are $m+1$ linearly independent vectors. Thus m can be at most $n-1$. □

3 Quantum Latin isometry squares and error detecting codes

We now introduce a generalisation of quantum Latin squares. A unit vector $|\Psi\rangle$ of dimension n is a trivial example of an isometry $|\Psi\rangle : \mathbb{C} \to \mathbb{C}^n$. We can thus consider n dimensional QLSs as arrays of isometries of this type. This perspective leads to the following definition which generalises QLSs.

Definition 19 (Isometric Latin square)**.** An n-by-n array of isometries $k_{ij} : \mathbb{C}^{a_{ij}} \to \mathbb{C}^d$ is an *isometric Latin square,* denoted (k_{ij}, a_{ij}, d) if the following hold for all $i, j, p, q \in \{0, \ldots, n\}$:

$$k^\dagger_{ip} \circ k_{iq} = \delta_{pq} \mathbb{I}_{a_{ip}} \tag{10}$$

$$k^\dagger_{pj} \circ k_{qj} = \delta_{pq} \mathbb{I}_{a_{mj}} \tag{11}$$

$$\sum_{i=0}^{n-1} k_{ij} \circ k^\dagger_{ij} = \sum_{j=0}^{n-1} k_{ij} \circ k^\dagger_{ij} = \mathbb{I}_d \tag{12}$$

Remark 20. Quantum Latin squares of size $n \times n$ are isometric Latin squares such that $a_{ij} = 1$ for all i and j, and are therefore of the form $(|\Phi_{ij}\rangle, 1, n)$.

We can build quantum Latin isometry squares from arbitrary families of unitaries, as follows.

Definition 21. For a Hilbert space $\mathbb{C}^n$ equipped with a family of m unitaries $\{U_i : \mathbb{C}^n \to \mathbb{C}^n | i \in [m]\}$, we can build a quantum Latin isometry square of size m as $(U_i \delta_{ij}, n\delta_{ij}, n)$.

Such a quantum Latin isometry square $L(\mathscr{U})$ is diagonal, with nonzero isometries only on the leading diagonal. It is straightforward to see that the quantum Latin isometry square axioms are satisfied.

3.1 Skew projective permutation matrices

Given a pair of isometric Latin squares that share the same multiset of values a_{ij} we can compose them in the following way to form a new structure.

Definition 22 (Skew projective permutation matrix)**.** Given a pair of isometric Latin squares (k_{ij}, a_{ij}, b) and (q_{ij}, a_{ij}, b) then we define the n-by-n array of linear operators $T_{ij} : \mathbb{C}^b \to \mathbb{C}^b$ as defined below to be a *skew projective permutation matrix* (skew PPM) T.

$$T_{ij} := q_{ij} \circ k^\dagger_{ij} \tag{13}$$

A *partial isometry* is a linear map such that the restriction to the orthogonal complement of its kernel is an isometry. Alternatively, a partial isometry A is a linear map such that $A \circ A^\dagger \circ A = A$. The *initial space* of a partial isometry is the orthogonal complement of its kernel. The *final space* is its range [4].

Given a pair of isometries $k_{ij}, q_{ij} : \mathbb{C}^{a_{ij}} \to \mathbb{C}^b$ we can form the following composite linear maps

$$K_{ij} : = k_{ij} \circ k^\dagger_{ij} \tag{14}$$

$$Q_{ij} : = q_{ij} \circ q^\dagger_{ij} \tag{15}$$

$$T_{ij} : = q_{ij} \circ k^\dagger_{ij} \tag{16}$$

It is easy to show that K_{ij} and Q_{ij} are orthogonal projectors. Since $T_{ij} \circ T^\dagger_{ij} \circ T_{ij} = q_{ij} \circ k^\dagger_{ij} \circ k_{ij} \circ q^\dagger_{ij} \circ q_{ij} \circ k^\dagger_{ij} = q_{ij} \circ k^\dagger_{ij} = T_{ij}$ we have that T_{ij} is a partial isometry. The initial and final spaces of T are the spaces projected onto by K_{ij} and Q_{ij} respectively.

By the above we can alternatively define skew PPMs as matrices of partial isometries $T_{ij} : \mathbb{C}^b \to \mathbb{C}^b$ such that the initial and final spaces are orthogonal and span $\mathbb{C}^b$ along each row and column.

Remark 23. Skew PPMs were shown to characterise quantum channels that preserve pointer states and are referred to as matrices of partial isometries fulfilling certain additional conditions in a recent paper by Benoist and Nechita (See [4], Definition 3.2 conditions (C1) to (C4)).

Projective permutation matrices (PPMs) are square arrays of projectors that form a projective POVM on every row and column. Skew PPMs generalise PPMs, which can be seen as skew PPMs coming from a pair of identical isometric Latin squares, in which case the partial isometries T_{ij} are all orthogonal projectors.

Remark 24. PPMs, also known as magic unitaries and quantum bijections between classical sets, have recently appeared in the context of quantum non-local games [1, 2, 12, 13] and the study of compact quantum groups [3, 5, 16].

We now extend the definition of orthogonal quantum Latin squares to isometric Latin squares.

Definition 25 (Orthogonal isometric Latin squares)**.** A pair of isometric Latin squares (k_{ij}, a_{ij}, d) and (q_{ij}, a_{ij}, d) are *orthogonal* if the operators $T_{ij} = q_{ij} \circ k_{ij}^{\dagger}$ are span the operator space and for all non-zero T_{ij} we have that $\mathrm{Tr}(T_{ij}^{\dagger} \circ T_{ij}) = a$ for some $a \in \mathbb{C}$.

We refer to a skew PPM composed of a pair of orthogonal isometric Latin squares as an *orthogonal skew PPM*.

We now give a more algebraic characterisation of the above definition.

Lemma 26. *Given a pair of $n \times n$ isometric Latin squares (k_{ij}, a_{ij}, d) and (q_{ij}, a_{ij}, d) define the following linear map $S : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^d \otimes \mathbb{C}^d$:*

$$S := \sum_{i,j,p,q=0}^{n-1} \sum_{x=0}^{d-1} |ij\rangle \otimes \langle x| q_{ij} \circ k_{ij}^{\dagger} \otimes \langle x| \tag{17}$$

The isometric Latin squares (k_{ij}, a_{ij}, d) and (q_{ij}, a_{ij}, d) are orthogonal just when S is an isometry.

Proof. Since T_{ij} span the operator space we have:

$$\sum_{i,j=0}^{n-1} \sum_{x,y=0}^{d-1} |y\rangle \otimes k_{ij} \circ q_{ij}^{\dagger} |y\rangle \langle x| q_{ij} \circ k_{ij}^{\dagger} \otimes \langle x| = \mathbb{I}_{d^2} \tag{18}$$

□

Remark 27. We also have that $\mathrm{Tr}(T_{ij}^{\dagger} \circ T_{pq}) = \delta_{ip}\delta_{jq}\mathrm{Tr}(T_{ij}^{\dagger} \circ T_{ij})$.

Remark 28. A PPM can never be orthogonal since the projectors T_{ij} of a PPM span the operator space for every row and column.

We now show that orthogonal isometric Latin squares generalise orthogonal QLSs (and therefore orthogonal Latin squares).

Lemma 29. *Pairs of QLS are orthogonal isometric Latin squares if and only if they are orthogonal quantum Latin squares.*

Proof. Consider a pair of n-by-n QLSs $(|k_{ij}\rangle, 1, n)$ and $(|q_{ij}\rangle, 1, n)$ such that they are orthogonal isometric Latin squares by Definition 25. By Lemma 26, either S is an isometry. Since S is a linear operator on a finite dimensional Hilbert space S is unitary. This yields the following equation:

$$\sum_{i,j,x,y=0}^{n-1} |k_{ij}\rangle\langle q_{ij}|x\rangle\langle y|q_{ij}\rangle\langle k_{ij}| \otimes |x\rangle\langle y| = \sum_{i,j=0}^{n-1} |k_{ij}\rangle\langle k_{ij}| \otimes |q_{ij}^{*}\rangle\langle q_{ij}^{*}| = \mathbb{I}_{n^2}$$

By Lemmas 5 and 7 this holds if and only if $|q_{ij}\rangle$ and $|k_{ij}\rangle$ are orthogonal QLS. □

Analogously to MOLS and MOQLS, we define *mutually orthogonal isometric Latin squares* to be sets of pairwise orthogonal isometric Latin squares.

We now prove the main result of this section, a construction of error correcting codes from orthogonal skew PPMs. In there famous 1997 paper Knill and Laflamme proved that certain one-to-three 4-valent tensors can be used as encoding maps for quantum codes that detect a single error. These tensors have only slightly less structure than perfect tensors.

Theorem 30. *[9] Given a three-to-one tensor $\langle E_{ijk}| : \mathbb{C}^a \to \mathbb{C}^b \otimes \mathbb{C}^c \otimes \mathbb{C}^d$, it is an encoding map that detects a single error if the following hold:*

$$\sum_{i=0}^{b-1}\sum_{j=0}^{c-1}\sum_{l,k=0}^{d-1} |E_{ijl}\rangle\langle E_{ijk}| \otimes |l\rangle\langle k| = \mathbb{I}_a \otimes \mathbb{I}_d \tag{19}$$

$$\sum_{i=0}^{b-1}\sum_{l,j=0}^{c-1}\sum_{k=0}^{d-1} |E_{ilk}\rangle\langle E_{ijk}| \otimes |l\rangle\langle j| = \mathbb{I}_a \otimes \mathbb{I}_c \tag{20}$$

$$\sum_{l,i=0}^{b-1}\sum_{j=0}^{c-1}\sum_{k=0}^{d-1} |E_{ljk}\rangle\langle E_{ijk}| \otimes |l\rangle\langle i| = \mathbb{I}_a \otimes \mathbb{I}_b \tag{21}$$

Theorem 31. *Given an n-by-n pair of orthogonal isometric Latin squares (k_{ij}, a_{ij}, d) and (q_{ij}, a_{ij}, d) skew PPM $T = \sum_{i,j=0}^{n-1} T_{ij} \otimes |i\rangle\langle j|$; the following one-to-three tensor is an encoding map that detects a single error:*

$$\langle T| := \sum_{i,j=0}^{n-1} |i\rangle \otimes q_{ij} \circ k_{ij}^{\dagger} \otimes |j\rangle \tag{22}$$

Proof. First we show equation (19), we have:

$$\sum_{x=0}^{d-1}\sum_{l,i,j=0}^{n-1} |T\rangle\langle T| \otimes |l\rangle\langle i| = \sum_{x=0}^{d-1}\sum_{l,i,j=0}^{n-1} |l\rangle \otimes k_{lj} \circ q_{lj}^{\dagger}|x\rangle\langle x|q_{ij} \circ k_{ij}^{\dagger} \otimes \langle i|$$

$$\overset{(11)}{=} \sum_{i,j=0}^{n-1} |i\rangle \otimes k_{ij} \circ k_{ij}^{\dagger} \otimes \langle i| \overset{(12)}{=} \mathbb{I}_d \otimes \mathbb{I}_n$$

Equation (21) can be derived from equations (10) and (12) similarly.

We now show equation (20).

$$\sum_{x,y=0}^{d-1}\sum_{i,j=0}^{n-1} |T\rangle\langle T| \otimes |y\rangle\langle x| = \sum_{x,y=0}^{d-1}\sum_{i,j=0}^{n-1} |y\rangle \otimes k_{ij} \circ q_{ij}^{\dagger}|y\rangle\langle x|q_{ij} \circ k_{ij}^{\dagger} \otimes \langle x|$$

$$\overset{(18)}{=} \mathbb{I}_{d^2}$$

□

The condition that a pair of isometric Latin squares are dimensionally compatible in order to form a skew PPM, together with the orthogonality condition puts heavy constraints upon the possible values for a_{ij} and d. It is not obvious that such structures, other than that of orthogonal QLSs should exist. We show in the next example that pairs of orthogonal isometric Latin squares do exist and even in low dimension.

Example 32. We present a pair of orthogonal Latin isometry squares Q and K and associated orthogonal skew PPM T. We have $n = 8$, $d = 4$ and $a_{ij} = 2$ or 0 for all $i, j \in \{0, \ldots, 7\}$. There are $d^2 = 16$ non-zero T_{ij} as required to span the operator space.

We fix the computational basis $|a\rangle, |b\rangle$ for $\mathbb{C}^2$ and $|0\rangle, |1\rangle, |2\rangle$ $|3\rangle$ for $\mathbb{C}^4$.
We present the first Latin isometry square Q:

$\vert 0\rangle\langle a\vert + \vert 1\rangle\langle b\vert$	$\vert 2\rangle\langle a\vert + \vert 3\rangle\langle b\vert$	0	0	0	0	0	0
$\vert 2\rangle\langle a\vert + \vert 3\rangle\langle b\vert$	$\vert 1\rangle\langle a\vert + \vert 0\rangle\langle b\vert$	0	0	0	0	0	0
0	0	$\vert 0\rangle\langle a\vert - \vert 1\rangle\langle b\vert$	$\vert 3\rangle\langle b\vert - \vert 2\rangle\langle a\vert$	0	0	0	0
0	0	$\vert 2\rangle\langle a\vert - \vert 3\rangle\langle b\vert$	$\vert 0\rangle\langle a\vert - \vert 1\rangle\langle b\vert$	0	0	0	0
0	0	0	0	$\vert 1\rangle\langle a\vert + \vert 2\rangle\langle b\vert$	$\vert 0\rangle\langle a\vert + \vert 3\rangle\langle b\vert$	0	0
0	0	0	0	$\vert 3\rangle\langle b\vert - \vert 0\rangle\langle a\vert$	$\vert 2\rangle\langle b\vert - \vert 1\rangle\langle a\vert$	0	0
0	0	0	0	0	0	$\vert 0\rangle\langle a\vert + \vert 2\rangle\langle b\vert$	$\vert 3\rangle\langle b\vert - \vert 1\rangle\langle a\vert$
0	0	0	0	0	0	$\vert 1\rangle\langle a\vert + \vert 3\rangle\langle b\vert$	$\vert 2\rangle\langle b\vert - \vert 0\rangle\langle a\vert$

Now we present the Latin isometry square K:

$\vert 0\rangle\langle a\vert + \vert 1\rangle\langle b\vert$	$\vert 2\rangle\langle a\vert + \vert 3\rangle\langle b\vert$	0	0	0	0	0	0
$\vert 2\rangle\langle a\vert + \vert 3\rangle\langle b\vert$	$\vert 0\rangle\langle a\vert + \vert 1\rangle\langle b\vert$	0	0	0	0	0	0
0	0	$\vert 0\rangle\langle a\vert + \vert 1\rangle\langle b\vert$	$\vert 3\rangle\langle a\vert + \vert 2\rangle\langle b\vert$	0	0	0	0
0	0	$\vert 2\rangle\langle a\vert + \vert 3\rangle\langle b\vert$	$\vert 1\rangle\langle a\vert + \vert 0\rangle\langle b\vert$	0	0	0	0
0	0	0	0	$\vert 2\rangle\langle a\vert + \vert 1\rangle\langle b\vert$	$\vert 3\rangle\langle a\vert + \vert 0\rangle\langle b\vert$	0	0
0	0	0	0	$\vert 3\rangle\langle a\vert + \vert 0\rangle\langle b\vert$	$\vert 2\rangle\langle a\vert + \vert 1\rangle\langle b\vert$	0	0
0	0	0	0	0	0	$\vert 2\rangle\langle a\vert + \vert 0\rangle\langle b\vert$	$\vert 3\rangle\langle a\vert + \vert 1\rangle\langle b\vert$
0	0	0	0	0	0	$\vert 3\rangle\langle a\vert + \vert 1\rangle\langle b\vert$	$\vert 2\rangle\langle a\vert + \vert 0\rangle\langle b\vert$

Finally we present the associated skew projective permutation matrix T with entries T_{ij}:

$\vert 0\rangle\langle 0\vert + \vert 1\rangle\langle 1\vert$	$\vert 3\rangle\langle 2\vert + \vert 2\rangle\langle 3\vert$	0	0	0	0	0	0
$\vert 2\rangle\langle 2\vert + \vert 3\rangle\langle 3\vert$	$\vert 1\rangle\langle 0\vert + \vert 0\rangle\langle 1\vert$	0	0	0	0	0	0
0	0	$\vert 0\rangle\langle 0\vert - \vert 1\rangle\langle 1\vert$	$\vert 3\rangle\langle 2\vert - \vert 2\rangle\langle 3\vert$	0	0	0	0
0	0	$\vert 2\rangle\langle 2\vert - \vert 3\rangle\langle 3\vert$	$\vert 0\rangle\langle 1\vert - \vert 1\rangle\langle 0\vert$	0	0	0	0
0	0	0	0	$\vert 2\rangle\langle 1\vert + \vert 1\rangle\langle 2\vert$	$\vert 3\rangle\langle 0\vert + \vert 0\rangle\langle 3\vert$	0	0
0	0	0	0	$\vert 3\rangle\langle 0\vert - \vert 0\rangle\langle 3\vert$	$\vert 2\rangle\langle 1\vert - \vert 1\rangle\langle 2\vert$	0	0
0	0	0	0	0	0	$\vert 2\rangle\langle 0\vert + \vert 0\rangle\langle 2\vert$	$\vert 3\rangle\langle 1\vert - \vert 1\rangle\langle 3\vert$
0	0	0	0	0	0	$\vert 3\rangle\langle 1\vert + \vert 1\rangle\langle 3\vert$	$\vert 2\rangle\langle 0\vert - \vert 0\rangle\langle 2\vert$

By Theorem 31, the following three-to-one 4-valent tensor is an encoding map that detects a single error:

$$\langle T| := \sum_{i,j=0}^{n-1} |i\rangle \otimes T_{ij} \otimes |j\rangle$$

3.2 Orthogonal quantum Latin isometry squares from unitary error bases

Here we recall the standard notion of unitary error basis, and show that they can be used to construct orthogonal pairs of quantum Latin isometry squares, which are not quantum Latin squares. Unitary error bases were introduced by Werner [17], and provide the basic data for quantum teleportation, dense coding and error correction procedures [8, 15, 17].

Definition 33. For a Hilbert space $\mathbb{C}^n$, a *unitary error basis* is a family of unitary operators $U_i : H \to H$ which span the space of operators, and which are orthogonal under the trace inner product:

$$\mathrm{Tr}(U_i \circ U_j^\dagger) = n\delta_{ij} \tag{23}$$

We now show that unitary error bases can be characterized in terms of orthogonality of quantum Latin isometry squares.

Theorem 34. *For a Hilbert space $\mathbb{C}^n$ and a family of unitaries $U_i : \mathbb{C}^n \to \mathbb{C}^n$, the following are equivalent:*

- *the quantum Latin isometry squares $(U_i\delta_{ij}, n\delta_{ij}, n)$ and $(\mathbb{I}_n\delta_{ij}, n\delta_{ij}, n)$ are orthogonal;*
- *the family $\{U_i | i \in [n^2]\}$ forms a unitary error basis.*

Proof. The orthogonality condition for the quantum Latin isometry squares unpacks to the requirement that the matrices U_i are orthogonal and span the space of operators; this is exactly the unitary error basis condition. □

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