Towards a cohomology invariant for contextuality

Giovanni Carù
Department of Computer Science
University of Oxford
giovanni.caru@cs.ox.ac.uk

This is an abstract of the paper **Towards a cohomology invariant for non-locality and contextuality**, available at http://www.cs.ox.ac.uk/files/9898/Cohomology_Invariant.pdf

1 Introduction

Non-locality and contextuality are key features of quantum mechanics, which have been proven to play a crucial rôle as a fundamental resource for quantum information and computation [6, 7]. Abramsky and Brandenburger have developed a general and unified description of these phenomena using sheaf theory [1]. The sheaf theoretic framework provides a rigorous topological description of contextuality, which perfectly conveys the idea of contextuality as a fundamental discrepancy between local consistency and global inconsistency [3]. In this framework, empirical models, which contain all the information concerning the outcomes of an ideal experiment, are represented as presheafs over the set of available measurements. Then, contextuality corresponds to the impossibility of extending the local sections of the empirical model presheaf to global ones. In recent work, this topological viewpoint has been further developed by taking advantage of sheaf cohomology, a widely used theory in algebraic topology, suited to study extendability of local features to global ones [2, 3]. Although this method has been proved to detect contextuality in a variety of well-studied empirical models, it does not constitute a complete invariant, as testified by a significant amount of false positives. Despite the appearance of other studies on the cohomological structure of contextuality [5, 8, 9], this issue has never been solved. In this paper, we present a complete cohomology invariant which is applicable to the vast majority of empirical models. There are strong indications that the invariant is in fact universal.

2 Background

We are interested in studying an ideal experimental scenario. Let X be a set of measurement labels, some of which may not be carried out simultaneously. We define the **measurement cover** $\mathscr{M} \subseteq \mathscr{P}(X)$ containing the **contexts**, i.e. the maximal sets of jointly performable measurements. We assume that each measurement $m \in X$ has outcomes in a set O_m. The triple $\Sigma := \langle X, \mathscr{M}, (O_m)_{m \in X} \rangle$ describes the **measurement scenario**. A scenario can be represented as an abstract simplicial complex whose vertices are elements of X and whose faces are constituted by jointly performable measurements.

By applying measurements in X on a physical system,[1] we observe, for each context, which joint outcomes are empirically possible for the measurements in the context . The collective information on the possible events constitutes an **empirical model**, and it can be described as a presheaf $\mathscr{S} : \mathscr{P}(X)^{op} \to \mathbf{Set}$ over the discrete space X satisfying some assumptions such as **no-signalling**. An empirical model can be represented as a **bundle** over the measurement simplicial complex, as shown in Figure 1.

[1] Here, by **physical system**, we mean any instance of a – possibly abstract – measurable entity. This could be a quantum state to which we apply quantum measurements, a logical proposition where we are interested in the truth value of its variables, a database with queries, etc. This level of generality allows to witness contextuality even outside the scope of quantum physics.

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A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	1	1	1
a_1	b_2	0	1	1	1
a_2	b_1	0	1	1	1
a_2	b_2	1	1	1	0

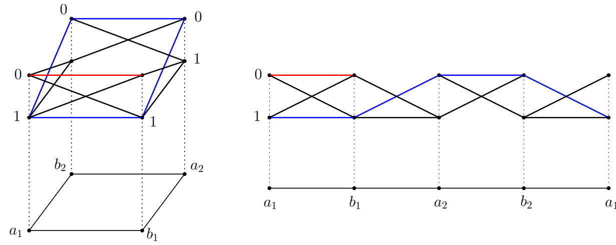

Figure 1: On the left, the possibility table of the Hardy model, defined on a bipartite scenario where Alice and Bob can choose between dichotomic measurements a_1, a_2 and b_1, b_2 respectively. On the right, the bundle representation of the model both in its 3-dimensional and planar version. At the base lies the measurement simplicial complex, and above each vertex is a fiber representing the possible outcomes for the corresponding measurement. The possible sections are represented by edges connecting fiber points. Global sections correspond to closed loops around the bundle, as shown in blue. One can verify that the red section s cannot be extended to a global one, which means that the model is contextual at s.

Possible joint outcomes for measurements in a context $C \in \mathscr{M}$ are described by **local sections** $s \in \mathscr{S}(C)$. On the other hand, deterministic assignments to all measurements in X are represented by **global sections** $g \in \mathscr{S}(X)$. In simple words, contextuality is the lack of a deterministic global explanation of a locally observed event. In the sheaf theoretic description, this corresponds to the impossibility of extending a local section to a global one. Figure 1 shows how this phenomenon can be visualised in bundle diagrams.

The question of extending local sections of a presheaf to global ones is well studied in algebraic topology. Sheaf cohomology constitutes the main tool at our disposal to tackle this problem, although this technique requires a presheaf of abelian groups. Following this idea, Abramsky et al. [2, 3] have developed a method to detect contextuality in empirical models based on Čech cohomology. The method consists of approximating the presheaf $\mathscr{S}$ by a presheaf of abelian groups $\mathscr{F} : \mathscr{P}(X)^{op} \to$ **AbGrp**, whose sections are formal linear combinations of sections of $\mathscr{S}$,[2] and deriving a **cohomology obstruction** to the extension of its local sections, obtaining a sufficient condition for contextuality.

False positives Despite this method being widely applicable, it does not constitute a complete invariant for contextuality. Indeed, there is a substantial number of false positives, even for the strongest forms of contextuality [5]. Such false positives are due to the fact that we are studying the cohomology of $\mathscr{F}$, an **abelian approximation** of $\mathscr{S}$ which admits a significantly higher number of global sections. Figure 2 gives a graphical intuition on how a false positive arises.

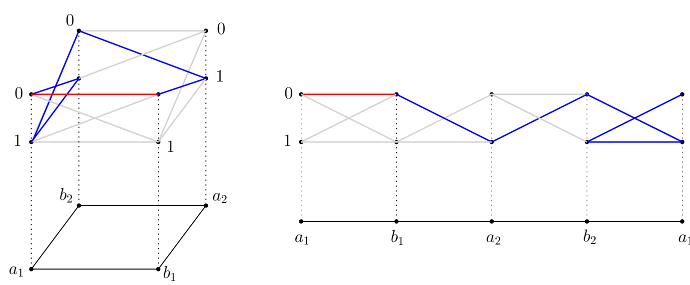

Figure 2: The Hardy model gives rise to a cohomological false positive. The model is contextual at the section marked in red, as shown in Figure 1. However, since $\mathscr{F}$ allows formal linear combination of sections, the blue loop – which is not a global section for $\mathscr{S}$ – is a valid global section of $\mathscr{F}$. For this reason, cohomology does not detect the contextuality of the model.

[2] Specifically, we define $\mathscr{F} := F_R \mathscr{S}$, where for all $U \subseteq X$, $F_R \mathscr{S}(U)$ is the free abelian group generated by the set $\mathscr{S}(U)$ on a semiring R.

3 Joint models and scenarios

In order to solve this problem, we adopt a different representation of the measurement structure. Given a scenario Σ we define the **first joint scenario** $\Sigma^{(1)} := \langle X^{(1)}, \mathscr{M}^{(1)}, (O_x^{(1)})_{x \in X^{(1)}} \rangle$, which carries the information concerning the intersectional properties of the contexts in $\mathscr{M}$. The simplicial complex of $\Sigma^{(1)}$ is 1-dimensional (i.e. it is a graph), with contexts of Σ as vertices, and edges between every pair of intersecting contexts, as shown in Figure 3. The outcomes for a vertex $C \in \mathscr{M}$ are all the theoretically possible **events** at C, i.e. $O_C^{(1)} := \prod_{m \in C} O_m$.

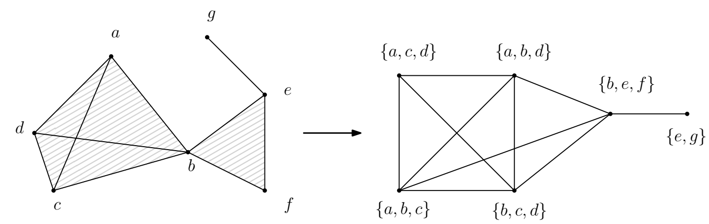

Figure 3: An example of a scenario and its joint model. The tetrahedron spanned by a, b, c, d is hollow.

Given an empirical model $\mathscr{S}$ on Σ, we define the **first joint model** $\mathscr{S}^{(1)} : \mathscr{P}(X^{(1)})^{op} \to$ **Set** on the first joint scenario. The local sections of $\mathscr{S}^{(1)}$ at each context in $\{C, C'\} \in \mathscr{M}^{(1)}$ are pairs of compatible sections $(s_C, s_{C'}) \in \mathscr{S}(C) \times \mathscr{S}(C')$. This feature allows us to immediately see all the different ways a section $s \in \mathscr{S}(C)$ can be extended to $\mathscr{S}(C \cup C')$. The key idea is to reiterate the joint model construction to understand when a section $s \in \mathscr{S}(C)$ can be extended to $\mathscr{S}(\bigcup_{C \in \mathscr{M}} C) = \mathscr{S}(X)$. For this reason, we let $\mathscr{S}^{(0)} := \mathscr{S}$, and define the k-**th joint model** $\mathscr{S}^{(k)} := \left(\mathscr{S}^{(k-1)}\right)^{(1)}$ for all $k \geq 1$.

4 A cohomology invariant for contextuality

It turns out that the successive application of the joint model construction eliminates cohomological false positives in an extremely vast class of scenarios. The first important result we present is the following:

Theorem 1. *Let Σ be a k-**cyclic scenario**, i.e. it is such that $\Sigma^{(1)}$ is a cycle of size k. Then, a model $\mathscr{S}$ on Σ is contextual at a section s if and only $\mathscr{S}^{(k-1)}$ is cohomologically contextual at s.*

This means that Čech cohomology gives rise to a full invariant for contextuality in cyclic scenarios, which solves all of the known false positives in the literature, e.g. the Hardy model, whose 3-rd joint model is pictured in Figure 4.

Cyclicity of the scenario is a fundamental aspect in the theory of contextuality. It has been shown in [4], via an adaptation of Vorob'ev's theorem [10], that the existence of a cycle in the first joint scenario is a necessary condition for contextuality. Thanks to this important result, we can extend Theorem 1 to the enormous class of models satisfying the **cyclic contextuality property (CCP)**, i.e. those models that display their contextuality on a cycle.

Theorem 2. *Let Σ be a general scenario. A model $\mathscr{S}$ on Σ is contextual at a section s if and only if $\mathscr{S}^{(N-1)}$ is cohomologically contextual at s. Here, N denotes the size of the contextual cycle of s.*

The class of CCP models is so vast that we could not produce an example of a ¬CCP model. Not only a hypothetical false positive for our refined method would have to be ¬CCP, but it would have to present a cohomological false positive in *all* its joint models. For this reason, we propose the following conjecture.

Conjecture 3. *Given a model $\mathscr{S}$ on a scenario Σ, there exists a $k \geq 0$ such that $\mathscr{S}$ is conextual at a section s if and only if $\mathscr{S}^{(k)}$ is cohomologically contextual at s.*

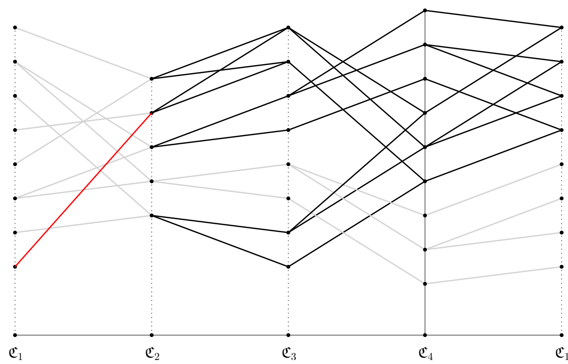

Figure 4: The third joint model of the Hardy model. The section in red 'represents' the original red section of Figure 1. We have highlighted in black all the possible attempts to extend it to a cohomological closed loop. We can clearly see that none of them succeeds.

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