Impossibility of mixed-state purification in any alternative to the Born Rule

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1 Introduction

The postulates of quantum theory can be divided into two sets: the dynamical postulates and the probabilistic postulates. The dynamical postulates determine the pure states, their dynamics and composition rules. The probabilistic postulates fix measurements outcomes and associated probabilities over pure states. The measurement postulates can be identified with the set of functions which give the probabilities of measurement outcomes. In quantum theory this is just the set of all functions of the form $F(\psi) = \langle\psi|\hat{F}|\psi\rangle$ where $\hat{F} \geq 0$. In this work we study systems described by the the same dynamical postulates as quantum theory, but different probabilistic ones. An alternative to the measurement postulates of quantum theory is just a set of functions $\{F(\psi)\}$ with images in $[0,1]$ for all rays ψ. We call these functions outcome probability functions (OPFs). In such an alternative measurement outcomes are no longer associated to positive semi-definite operators, and probabilities of outcomes are no longer given by the Born rule.

In previous work [1] we considered single systems with the same dynamical postulates as quantum systems and classified all possible probabilistic postulates. All these postulates (sets of outcome probability functions) were shown to be in correspondence with representations of the unitary group.

In the present work we extend this study of alternative measurement postulates by considering composite systems. We show that operational features of composition lead to additional constraints on the allowed sets of OPFs. We then study the informational and operational properties of these alternative theories. We show that all theories with the same dynamical postulates as quantum theory but different probabilistic ones must violate local tomography and purification.

This work has two main motivations. Previous suggested modified Born rules were more restricted than the ones introduced in this work, and were shown to be non-signalling [2, 3]. In this work we show that by considering more general modifications it is possible to obtain non-signalling bi-partite systems. A second motivation comes from the desire to find foil theories, that is non-classical theories which can be compared to quantum theory. We then study the compositional properties of this alternative theories, and discuss them in light of existing work in general probabilistic theories.

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2 Operational constraints on measurement postulates

In a first part we discuss how certain features of operational theories, such as measuring a system using an ancillary system, impose constraints on the allowed sets of outcome probability functions (alternative measurement postulates). We call $\mathscr{F}_d$ the set of outcome probability functions of a system $\mathbb{C}^d$. For two systems $\mathbb{C}^{d_\mathsf{A}}$, $\mathbb{C}^{d_\mathsf{B}}$ to compose to a third system $\mathbb{C}^{d_\mathsf{A} d_B}$ (where each system has associated sets of outcome probability functions $\mathscr{F}_{d_\mathsf{A}}$, $\mathscr{F}_{d_\mathsf{B}}$ and $\mathscr{F}_{d_\mathsf{A} d_\mathsf{B}}$) the associated sets $\mathscr{F}_{d_\mathsf{A}}$, $\mathscr{F}_{d_\mathsf{B}}$ and $\mathscr{F}_{d_\mathsf{A} d_\mathsf{B}}$ need to meet several criteria. For example for every OPF F_A in $\mathscr{F}_{d_\mathsf{A}}$ and every OPF F_B in $\mathscr{F}_{d_\mathsf{B}}$ there should exist an OPF in $\mathscr{F}_{d_\mathsf{A} d_\mathsf{B}}$ corresponding to the joint measurement outcome on the composite system. We list all the conditions that operationality imposes on the sets of OPFs. We translate the requirement of existence of joint local measurements into a representation theoretic feature, and show that all systems with pure states $\mathbb{C}^9$ have representations which meet have this feature. This shows they meet a necessary (but not sufficient) criterion for being the composites of two $\mathbb{C}^3$ systems (with alternative measurement postulates). We provide a specific toy model of systems with alternative Born rules which compose. This toy theory is a subtheory of quantum theory which violates purification and local tomography as well as having restricted effects, all of which are noisy. We show that in a second part that all theories with alternative Born rules violate both local tomography and purification.

3 Properties of alternative theories

Local tomography is a property of operational theories which states that full tomography of a bi-partite system can be carried out using joint local measurements only. We show that all systems which are locally tomographic must have a certain representation theoretic feature expressed in terms of a branching rule. A branching rule determines how a given representation decomposes into subrepresentations when restricted to a subgroup. We show that all $\mathbb{C}^9$ systems we study have associated representations which do not have the branching structure required by local tomography. This shows that all $\mathbb{C}^9$ systems with alternative measurement postulates violate local tomography.

Purification is another common property in GPTs which states that all mixed states of a system can be obtained as the reduction of a pure state of a larger system. We show that all the alternative theories in this work violate purification, which entails that in all these alternative theories there exist mixed states which cannot be expressed as the reduction of some larger pure state.

4 Discussion and future work

Our results show that one could derive, or single out the quantum measurement postulates from the requirement of local tomography or purification. These principles are operational/informational. Previous attempts to derive the Born rule used decision theory [11] or symmetry arguments [12]. We discuss our results in light of recent work such as [13], where the authors show that any successor theory to quantum theory which decoheres to quantum theory (subject to some requirements) must violate purification. We also argue that the requirement of local tomography, which is often deemed natural in many approaches, seems like a rare feature in general theories. Indeed we find that arbitrarily small modifications to the measurement postulates of quantum theory lead to a violation of local tomography. Many general results have been derived in the framework of GPTs showing features or properties of various natural families of theories which either obey the principle of local tomography [4, 5, 6] or the purification principle

[7, 8, 9, 10]. These results do not apply to the large family of theories in this work. This shows that assuming local tomography or purification may be unduly restrictive when studying general probabilistic theories.

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