Beyond the Cabello-Severini-Winter framework: a hypergraph-theoretic framework for Spekkens contextuality

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A key concern in the foundations of quantum theory, as well as in applications such as quantum information and computation, is the question of what phenomena count as genuinely nonclassical, i.e., as admitting no classical description, for some precise notion of "classicality". Bell's notion of local causality, for example, provides a very compelling notion of classicality, one often formulated in modern terminology as constraints (Bell inequalities) on the strength of correlations established by pre-shared randomness between non-communicating parties. However, while this is a very pertinent notion of nonclassicality in situations that involve spacelike separated parties doing some measurements (or at the very least, some other guarantee that the parties are indeed not signalling to each other), it's not the most relevant notion for situations where the correlations are between measurements carried out in the same lab (or even on the same quantum system), rather than non-signalling labs. This latter situation is the one most commonly encountered in most experiments that are not testing Bell inequality violations, but where one still seeks a notion of nonclassicality to benchmark the "quantumness" of one's devices. Such situations (of the non-Bell type) are native to the Kochen-Specker (KS) theorem (and the attendant notion of KS-contextuality as nonclassicality), and a recent slew of results arguing for the necessity of KS-contextuality in some models of quantum computing – such as measurement-based quantum computation (MBQC) [1] and quantum computation with state injection [2, 3, 4] – bear witness to the relevance of the Kochen-Specker theorem for quantum computing.

However, a crucial conceptual hurdle is that the Kochen-Specker theorem in its traditional formulation is not experimentally testable in finite-precision experiments because the set of projectors (in any given Hilbert space) that do not admit a KS contradiction is dense in the set of all projectors (which includes, in particular, those that do yield a KS contradiction) [5]. Since one expects noise in any real-world implementation of a quantum experiment, whether it's in quantum computing or some other task, it's not clear how the theoretical relevance of the Kochen-Specker theorem for quantum computing, say, bears on its practical relevance for any real-world implementation. Indeed, a major issue with the KS theorem, and the associated violation of Bell-Kochen-Specker inequalities in statistical proofs of the KS theorem, is that no account is taken of the possibility of noise in the measurements, i.e., they seem to suggest the presence of nonclassicality even if the measurements are very noisy, if one tries to apply them to the case of nonprojective POVMs. To avoid running into pathologies having to do with trivial POVMs (where all POVM elements are proportional to identity), most traditional treatments restrict the application of these inequalities to projective measurements. However, a good notion of nonclassicality should continue to apply even when some noise is allowed, rather than suddenly becoming inapplicable the moment one's measurements are slightly nonprojective. It should specify criteria for noise thresholds below which nonclassicality can be witnessed, so that there is a quantitative accounting of how far away from the theoretical ideal (e.g., projective measurements) one can be while still witnessing some nonclassicality; this must be particularly so if the goal is to exploit this nonclassicality in concrete applications such as in quantum computation or information. This is exactly what is achieved by the noise-robust notion of contextuality in Ref. [6], which we use in this submission [7] to obtain criteria for witnessing nonclassicality in the presence of noise, based on our approach to operationalizing the Kochen-Specker theorem [8, 9]. Combined with earlier methods for dealing with limitations such as the finite-precision of a real experiment [10] within the generalized probabilistic theory (GPT) framework, our work provides noise-robust witnesses of nonclassicality that work for any GPT, including quantum theory.

Based on the general method introduced in Ref. [8], our submission [7] shows how to turn the graph-theoretic approach due to Cabello, Severini, and Winter (CSW) [11] for KS-contextuality to a hypergraph-theoretic approach for Spekkens contextuality. In the process, we introduce a new hypergraph invariant — the *weighted max-predictability* $\beta(\Gamma_G, q)$ for a contextuality scenario Γ_G with hyperedges weighted by probability distribution $q \equiv \{q_e\}_{e\in E(\Gamma_G)}$ — that is essential in our approach to contextuality but has no counterpart in the CSW framework. At the same time, we also incorporate the graph invariants of CSW in our noise-robust noncontextuality inequalities that reduce, in an ideal limit, to the bounds on KS-noncontextuality from the CSW framework. Our submission thus bridges the gap between traditional approaches to KS contextuality and our operational approach to contextuality [6], incorporating the former into our generalized framework. The general form of any noise-robust noncontextuality inequality in our framework is the following:

$$R \overset{\text{NC}}{\leq} \alpha(G,w) + \frac{\alpha_*(G,w) - \alpha(G,w)}{p_*} \frac{1 - \text{Corr}}{1 - \beta(\Gamma_G, q)}, \tag{1}$$

where the graph invariants $\alpha(G,w)$ and $\alpha_*(G,w)$ are the independence number and the fractional packing number of a vertex-weighted graph, G, as defined in Ref. [11]. The hypergraph invariant $\beta(\Gamma_G, q)$ is the weighted max-predictability we need to define to accommodate contextuality à la Spekkens [6] in our framework. The quantity Corr quantifies the source-measurement correlations that account for noise in the measurements. In the ideal (noiseless) case, we have Corr = 1 and we recover the KS-noncontextuality bound of $R \leq \alpha(G, w)$, as in Ref. [11].

Together with previous work [9, 12, 8] and some upcoming work [13], this completes the project of handling Kochen-Specker type scenarios within the Spekkens approach to contextuality.

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