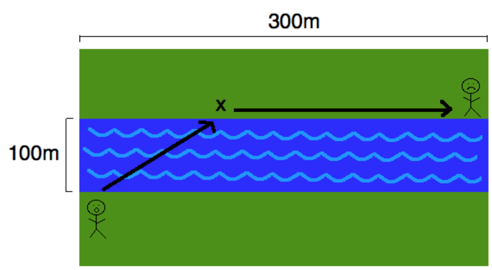
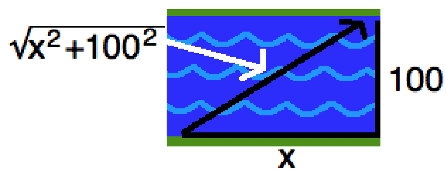
Full solution to the swimming / running optimization problem
(Last example given in class on Wednesday, July 28).

Example: You are standing on the bank of a river that is 100m wide, and see someone needing help 300m up the opposite shore. You can swim at 3m/s and run at 5m/s, and you want to get to the person as quickly as possible. To what point on the opposite shore should you swim, before running the rest of the way?

Solution:

(1)–(3) Let x be the distance up along the opposite shore where I get out of the river and start running. Let T be the total time, which is to be minimized.





(4) T = swim time + run time. Recall that time $= \frac{\text{distance}}{\text{speed}}$. How far do I swim?

From the picture we see the swim distance is $\sqrt{x^2+100^2}$. Running distance is $300 - x$. Dividing by the speeds for each distance, this gives

$$T = \frac{\sqrt{x^2+100^2}}{3} + \frac{300-x}{5}.$$

(5) The expression for T is already in terms of only one variable.

(6) $T' = \frac{1}{3}\left(\frac{2x}{2\sqrt{x^2+100^2}}\right) - \frac{1}{5} = \frac{5x - 3\sqrt{x^2+100^2}}{3\sqrt{x^2+100^2}}$. $\quad T' = 0 \quad \Rightarrow \quad 5x = 3\sqrt{x^2+100^2}$
$\Rightarrow \quad 25x^2 = 9x^2 + 9(100^2) \quad \Rightarrow \quad 16x^2 = 9(100^2) \quad \Rightarrow \quad 4x = 3(100) \quad \Rightarrow \quad x = 75.$
So $x = 75$ is a critical number; $T'(70) < 0$ and $T'(100) > 0$ so T is decreasing until $x = 75$ and increasing after, which means there is a relative minimum at $x = 75$. Now, x can be any value in the closed interval $[0, 300]$. So we must check to see if one of the endpoints 0 or 300 is the *absolute* minimum.

$$T(0) = \frac{100}{3} + \frac{300}{5} \approx 93.3, \quad T(75) \approx 66.6, \quad T(300) \approx 105.4,$$

so the shortest time (absolute minimum of T) is 66.6 seconds, which occurs when $x = 75$. Thus, I should swim to a point 75m up the opposite shore and then run the rest of the way.