Two-layered logics for paraconsistent probabilities*

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Abstract. We discuss two two-layered logics formalising reasoning with paraconsistent probabilities that combine the Łukasiewicz $[0,1]$-valued logic with Baaz $\triangle$ operator and the Belnap–Dunn logic. The first logic $\mathsf{Pr}^{\mathsf{Ł}^2}_{\triangle}$ (introduced in [7]) formalises a 'two-valued' approach where each event ϕ has independent positive and negative measures that stand for, respectively, the likelihoods of ϕ and $\neg\phi$. The second logic $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_{\triangle}}$ that we introduce here corresponds to 'four-valued' probabilities. There, ϕ is equipped with four measures standing for pure belief, pure disbelief, conflict and uncertainty of an agent in ϕ.

We construct faithful embeddings of $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_{\triangle}}$ and $\mathsf{Pr}^{\mathsf{Ł}^2}_{\triangle}$ into one another and axiomatise $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_{\triangle}}$ using a Hilbert-style calculus. We also establish the decidability of both logics and provide complexity evaluations for them using an expansion of the constraint tableaux calculus for $\mathsf{Ł}$.

Keywords: two-layered logics · Łukasiewicz logic · non-standard probabilities · paraconsistent logics · constraint tableaux

1 Introduction

Classical probability theory studies probability measures: maps from a probability space to $[0,1]$ that satisfy the (finite or countable) additivity[4] condition:

$$\mu\left(\bigcup_{i\in I} E_i\right) = \sum_{i\in I}\mu(E_i) \qquad (\forall i,j\in I : i\neq j \Rightarrow E_i\cap E_j=\varnothing)$$

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[4] In this paper, when dealing with the classical probability measures we will assume that they are *finitely* additive.

Above, the disjointness of E_i and E_j can be construed as their incompatibility. Most importantly, if a propositional formula ϕ is associated with an event (and interpreted as a statement about it), then ϕ and $\neg\phi$ are incompatible and $\phi \vee \neg\phi$ exhausts the entire sample space.

Paraconsistent probability theory, on the other hand, assumes that the probability measure of an event represents not the likelihood of it happening but an agent's certainty therein which they infer from the information given by the sources. As a *single* source can give incomplete or contradictory information, it is reasonable to assume that a 'contradictory' event $\phi \wedge \neg\phi$ can have a positive probability and that $\phi \vee \neg\phi$ does not necessarily have probability 1.

Thus, a logic describing events should allow them to be both true and false (if the source gives contradictory information) or neither true nor false (when the source does not give information). Formally, this means that $\neg$ does not correspond to the complement in the sample space.

Paraconsistent probabilities in BD The simplest logic to represent reasoning about information provided by sources is the Belnap–Dunn logic [15,4,3]. Originally, BD was presented as a four-valued propositional logic in the $\{\neg, \wedge, \vee\}$ language. The values represent the different accounts a source can give regarding a statement ϕ:

- **T** stands for 'the source only says that ϕ is true';
- **F** stands for 'the source only says that ϕ is false';
- **B** stands for 'the source says both that ϕ is false and that ϕ is true';
- **N** stands for 'the source does not say that ϕ is false nor that it is true'.

The interpretation of the truth values allows for a reformulation of BD semantics in terms of *two classical but independent valuations*. Namely,

	is true when	**is false when**
$\neg\phi$	ϕ is false	ϕ is true
$\phi_1 \wedge \phi_2$	ϕ_1 and ϕ_2 are true	ϕ_1 is false or ϕ_2 is false
$\phi_1 \vee \phi_2$	ϕ_1 is true or ϕ_2 is true	ϕ_1 and ϕ_2 are false

It is easy to see that there are no universally true nor universally false formulas in BD. Thus, BD satisfies the desiderata outlined above.

The original interpretation of the Belnapian truth values is given in terms of the information one has. However, the information is assumed to be crisp. Probabilities over BD were introduced to formalise situations where one has access to probabilistic information. For instance, a first source could tell that p is true with probability 0.4 and a second that p is false with probability 0.7. If one follows BD and treats positive and negative evidence independently, one needs a non-classical notion of probabilities to represent this information.

The first representation of paraconsistent probabilities in terms of BD was given in [16], however, no axiomatisation was provided. Dunn proposes to divide the sample space into four exhaustive and mutually exclusive parts depending on the Belnapian value of ϕ. An alternative approach was proposed in [26]. There, the authors propose two equivalent interpretations based on the two formulations of semantics. The first option is to give ϕ two *independent probability measures*: the one determining the likelihood of ϕ to be true and the other the likelihood of ϕ to be false. The second option follows Dunn and also divides the sample space according to whether ϕ has value **T**, **B**, **N**, or **F** in a given state. Note that in both cases, the probabilities are interpreted *subjectively*.

The main difference between these two approaches is that in [16], the probability of $\phi \wedge \phi'$ is entirely determined by those of ϕ and ϕ' which makes it compositional. On the other hand, the paraconsistent probabilities proposed in [26] are not compositional w.r.t. conjunction. In this paper, we choose the latter approach since it can be argued [14] that belief is not compositional.

A similar approach to paraconsistent probabilities can be found in, e.g. [9,29]. There, probabilities are defined over an extension of BD with classicality and non-classicality operators. It is worth mentioning that the proposed axioms of probability are very close to those from [26]: e.g., both allow measures $\mathtt{p}$ s.t. $\mathtt{p}(\phi)+\mathtt{p}(\neg\phi) < 1$ (if the information regarding ϕ is incomplete) or $\mathtt{p}(\phi)+\mathtt{p}(\neg\phi) > 1$ (when the information is contradictory).

Two-layered logics for uncertainty Reasoning about uncertainty can be formalised via modal logics where the modality is interpreted as a measure of an event. The concrete semantics of the modality can be defined in two ways. First, using a modal language with Kripke semantics where the measure is defined on the set of states as done in, e.g., [19,12,13] for qualitative probabilities and in [11] for the quantitative ones. Second, employing a two-layered formalism (cf. [18,17], [2], and [7,6] for examples). There, the logic is split into two levels: the inner layer describes events, and the outer layer describes the reasoning with the measure defined on events. The measure is a *non-nesting* modality $\mathtt{M}$, and the outer-layer formulas are built from 'modal atoms' of the form $\mathtt{M}\phi$ with ϕ being an inner-layer formula. The outer-layer formulas are then equipped with the semantics of a fuzzy logic that permits necessary operations (e.g., Łukasiewicz for the quantitative reasoning and Gödel for the qualitative).

In this work, we choose the two-layered approach. First, it is more modular than the usual Kripke semantics: as long as the logic of the event description is chosen, we can define different measures on top of it using different upper-layer logics. Second, the completeness proof is very simple since one only needs to translate the axioms of the given measure into the outer-layer logic. Finally, even though, the traditional Kripke semantics is more expressive than two-layered logics, this expressivity is not really necessary in many contexts. Indeed, people rarely say something like 'it is probable that it is probable that ϕ'. Moreover, it is considerably more difficult to motivate the assignment of truth values in the nesting case, in particular, when one and the same measure is applied both to a propositional and modalised formula as in, e.g., $\mathtt{M}(p \wedge \mathtt{M}q)$.

We will also be dealing with the formalisation of the *quantitative* probabilistic reasoning. Formally, this means that we assume that the agents can assign numerical values to their certainty in a given proposition or say something like 'I am twice as certain that it is going to rain than that it is going to snow'. Thus, we need a logic that can express the paraconsistent counterparts of the additivity condition as well as basic arithmetic operations. We choose the Łukasiewicz logic (Ł) for the outer layer since it can define (truncated) addition and subtraction on $[0,1]$.

Plan of the paper Our paper continues the project proposed in [8] and continued in [7] and [6]. Here, we set to provide a logic that formalises the reasoning with four-valued probabilities as presented in [26]. The rest of the text is organised as follows. In Section 2, we recall two approaches to probabilities over BD from [26]. In Section 3, we provide the semantics of our two-layered logics and in Section 4, we axiomatise them using Hilbert-style calculi. In Section 5, we prove that all our logics are decidable and establish their complexity evaluations. Finally, we wrap up our results in Section 6.

2 Two approaches to paraconsistent probabilities

We begin with defining the semantics of BD on sets of states. The language of BD is given by the following grammar (with $\texttt{Prop}$ being a countable set of propositional variables).

$$\mathcal{L}_{\mathsf{BD}} \ni \phi \coloneqq p \in \texttt{Prop} \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi)$$

Convention 1 *In what follows, we will write* $\texttt{Prop}(\phi)$ *to denote the set of variables occurring in* ϕ *and* $\mathsf{Lit}(\phi)$ *to denote the set of* literals *(i.e., variables or their negations) occurring in* ϕ*. Moreover, we use* $\texttt{Sf}(\phi)$ *to stand for the set of all subformulas of* ϕ*.*

We are also going to use two kinds of formulas: the single- and the two-layered ones. To make the differentiation between them simpler, we use Greek letters from the end of the alphabet (ϕ*,* χ*,* ψ*, etc.) to designate the first kind and the letters from the beginning of the alphabet (*α*,* β*,* γ*, ...) for the second kind.*

Furthermore, we use v *(with indices) to stand for the valuations of single-layered formulas and* e *(with indices) for the two-layered formulas.*

Definition 1 (Set semantics of BD). *Let* $\phi, \phi' \in \mathcal{L}_{\mathsf{BD}}$*,* $W \neq \varnothing$*, and* $v^+, v^- : \texttt{Prop} \rightarrow 2^W$*. For a model* $\mathfrak{M} = \langle W, v^+, v^- \rangle$*, we define notions of* $w \vDash^+ \phi$ *and* $w \vDash^- \phi$ *for* $w \in W$ *as follows.*

$$\begin{array}{ll} w \vDash^+ p \text{ iff } w \in v^+(p) & w \vDash^- p \text{ iff } w \in v^-(p) \\ w \vDash^+ \neg\phi \text{ iff } w \vDash^- \phi & w \vDash^- \neg\phi \text{ iff } w \vDash^+ \phi \\ w \vDash^+ \phi \wedge \phi' \text{ iff } w \vDash^+ \phi \text{ and } w \vDash^+ \phi' & w \vDash^- \phi \wedge \phi' \text{ iff } w \vDash^- \phi \text{ or } w \vDash^- \phi' \\ w \vDash^+ \phi \vee \phi' \text{ iff } w \vDash^+ \phi \text{ or } w \vDash^+ \phi' & w \vDash^- \phi \vee \phi' \text{ iff } w \vDash^- \phi \text{ and } w \vDash^- \phi' \end{array}$$

We denote the positive and negative extensions of a formula as follows:

$$|\phi|^+ \coloneqq \{w \in W \mid w \vDash^+ \phi\} \qquad |\phi|^- \coloneqq \{w \in W \mid w \vDash^- \phi\}.$$

We say that a sequent $\phi \vdash \chi$ *is* valid on $\mathfrak{M} = \langle W, v^+, v^- \rangle$ *(denoted,* $\mathfrak{M} \models [\phi \vdash \chi]$*) iff* $|\phi|^+ \subseteq |\chi|^+$ *and* $|\chi|^- \subseteq |\phi|^-$*. A sequent* $\phi \vdash \chi$ *is* BD-valid *(*$\phi \vDash_{\mathsf{BD}} \chi$*) iff it is valid on every model. In this case, we will say that* ϕ entails χ*.*

Now, we can use the above semantics to define probabilities on the models. We adapt the definitions from [26].

Definition 2 (BD models with $\pm$-probabilities). *A* BD model with a $\pm$-probability *is a tuple* $\mathfrak{M}_\mu = \langle \mathfrak{M}, \mu \rangle$ *with* $\mathfrak{M}$ *being a* BD *model and* $\mu : 2^W \rightarrow [0, 1]$ *satisfying:*

mon: *if* $X \subseteq Y$*, then* $\mu(X) \leq \mu(Y)$*;*
neg: $\mu(|\phi|^-) = \mu(|\neg\phi|^+)$*;*
ex: $\mu(|\phi \vee \chi|^+) = \mu(|\phi|^+) + \mu(|\chi|^+) - \mu(|\phi \wedge \chi|^+)$*.*

To facilitate the presentation of the four-valued probabilities defined over BD models, we introduce additional extensions of ϕ defined via $|\phi|^+$ and $|\phi|^-$.

Convention 2 *Let* $\mathfrak{M} = \langle W, v^+, v^- \rangle$ *be a* BD *model,* $\phi \in \mathcal{L}_{\mathsf{BD}}$*. We set*

$$|\phi|^{\mathsf{b}} = |\phi|^+ \setminus |\phi|^- \qquad |\phi|^{\mathsf{d}} = |\phi|^- \setminus |\phi|^+$$
$$|\phi|^{\mathsf{c}} = |\phi|^+ \cap |\phi|^- \qquad |\phi|^{\mathsf{u}} = W \setminus (|\phi|^+ \cup |\phi|^-)$$

We call these extensions, respectively, pure belief*,* pure disbelief*,* conflict*, and* uncertainty in ϕ*, following [26].*

Definition 3 (BD models with 4-probabilities). *A* BD model with a **4**-probability *is a tuple* $\mathfrak{M}_{\mathbf{4}} = \langle \mathfrak{M}, \mu_{\mathbf{4}} \rangle$ *with* $\mathfrak{M}$ *being a* BD *model and* $\mu_{\mathbf{4}} : 2^W \rightarrow [0, 1]$ *satisfying:*

part: $\mu_{\mathbf{4}}(|\phi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\phi|^{\mathsf{d}}) + \mu_{\mathbf{4}}(|\phi|^{\mathsf{u}}) + \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}}) = 1$*;*
neg: $\mu_{\mathbf{4}}(|\neg\phi|^{\mathsf{b}}) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{d}})$*,* $\mu_{\mathbf{4}}(|\neg\phi|^{\mathsf{c}}) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}})$*;*
contr: $\mu_{\mathbf{4}}(|\phi \wedge \neg\phi|^{\mathsf{b}}) = 0$*,* $\mu_{\mathbf{4}}(|\phi \wedge \neg\phi|^{\mathsf{c}}) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}})$*;*
BCmon: *if* $\mathfrak{M} \models [\phi \vdash \chi]$*, then* $\mu_{\mathbf{4}}(|\phi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}}) \leq \mu_{\mathbf{4}}(|\chi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\chi|^{\mathsf{c}})$*;*
BCex: $\mu_{\mathbf{4}}(|\phi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}}) + \mu_{\mathbf{4}}(|\psi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\psi|^{\mathsf{c}}) = \mu_{\mathbf{4}}(|\phi \wedge \psi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\phi \wedge \psi|^{\mathsf{c}}) + \mu_{\mathbf{4}}(|\phi \vee \psi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\phi \vee \psi|^{\mathsf{c}})$*.*

Convention 3 *We will further utilise the following naming convention:*

- *we use the term* '$\pm$-probability' *to stand for* μ *from Definition 2;*
- *we call* $\mu_{\mathbf{4}}$ *from Defintion 3 a* '**4**-probability' *or a* 'four-valued probability'.

Recall that $\pm$-probabilities are referred to as 'non-standard' in [26] and [7]. As this term is too broad (four-valued probabilities are not 'standard' either), we use a different designation.

Let us quickly discuss the measures defined above. First, observe that $\mu(|\phi|^+)$ and $\mu(|\phi|^-)$ are independent from one another. Thus, μ gives two measures to each ϕ, as desired. Second, recall [26, Theorems 2–3] that every **4**-probability on a BD model induces a $\pm$-probability and vice versa. In the following sections, we will define two-layered logics for BD models with $\pm$- and **4**-probabilities and show that they can be faithfully embedded into each other.

Remark 1. Note, that for every BD model with a $\pm$-probability $\langle W, v^+, v^-, \mu \rangle$ (resp., BD model with **4**-probability $\langle W, v^+, v^-, \mu_{\mathbf{4}} \rangle$), there exist a BD model $\langle W', v'^+, v'^-, \pi \rangle$ with a *classical* probability measure π s.t. $\pi(|\phi|^+) = \mu(|\phi|^+)$ (resp., $\pi(|\phi|^{\mathsf{x}}) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{x}})$ for $\mathsf{x} \in \{\mathsf{b}, \mathsf{d}, \mathsf{c}, \mathsf{u}\}$) [26, Theorems 4–5]. Thus, we can further assume w.l.o.g. that μ and $\mu_{\mathbf{4}}$ are *classical probability measures* on W.

3 Logics for paraconsistent probabilities

In this section, we provide logics that are (weakly) complete w.r.t. BD models with $\pm$- and $\mathbf{4}$-probabilities. Since conditions on measures contain arithmetic operations on $[0,1]$, we choose an expansion of Łukasiewicz logic, namely, Łukasiewicz logic with $\triangle$ ($\mathsf{Ł}_\triangle$), for the outer layer. Furthermore, $\pm$-probabilities work with both positive and negative extensions of formulas, whence it seems reasonable to use $\mathsf{Ł}^2$ — a paraconsistent expansion of $\mathsf{Ł}$ (cf. [8,5] for details) with two valuations — v_1 (support of truth) and v_2 (support of falsity) — on $[0,1]$. This was done in [7] — the resulting logic $\mathsf{Pr}^{\mathsf{Ł}^2}_\triangle$ was proven to be complete w.r.t. BD models with $\pm$-probabilities.

We begin by recalling the language and standard semantics of Łukasiewicz logic with $\triangle$ and its paraconsistent expansion $\mathsf{Ł}^2_\triangle$.

Definition 4. *The standard* $\mathsf{Ł}_\triangle$*-algebra is a tuple* $\langle[0,1],\sim_{\mathsf{Ł}},\triangle_{\mathsf{Ł}},\wedge_{\mathsf{Ł}},\vee_{\mathsf{Ł}},\rightarrow_{\mathsf{Ł}},\odot_{\mathsf{Ł}},\oplus_{\mathsf{Ł}},\ominus_{\mathsf{Ł}}\rangle$ *with the operations are defined as follows.*

$$\sim_{\mathsf{Ł}}a\coloneqq 1-a \qquad\qquad \triangle_{\mathsf{Ł}}a\coloneqq\begin{cases}1 & \text{if } a=1\\ 0 & \text{otherwise}\end{cases}$$

$$a\wedge_{\mathsf{Ł}}b\coloneqq\min(a,b) \qquad a\vee_{\mathsf{Ł}}b\coloneqq\max(a,b) \qquad a\rightarrow_{\mathsf{Ł}}b\coloneqq\min(1,1-a+b)$$
$$a\odot_{\mathsf{Ł}}b\coloneqq\max(0,a+b-1) \qquad a\oplus_{\mathsf{Ł}}b\coloneqq\min(1,a+b) \qquad a\ominus_{\mathsf{Ł}}b\coloneqq\max(0,a-b)$$

Definition 5 (Łukasiewicz logic with $\triangle$). *The language of* $\mathsf{Ł}_\triangle$ *is given via the following grammar*

$\mathcal{L}_{\mathsf{Ł}}\ni\phi\coloneqq p\in\mathtt{Prop}\mid\sim\phi\mid\triangle\phi\mid(\phi\wedge\phi)\mid(\phi\vee\phi)\mid(\phi\rightarrow\phi)\mid(\phi\odot\phi)\mid(\phi\oplus\phi)\mid(\phi\ominus\phi)$
We will also write $\phi\leftrightarrow\chi$ *as a shorthand for* $(\phi\rightarrow\chi)\odot(\chi\rightarrow\phi)$.

A valuation is a map $v:\mathtt{Prop}\rightarrow[0,1]$ *that is extended to the complex formulas as expected:* $v(\phi\circ\chi)=v(\phi)\circ_{\mathsf{Ł}}v(\chi)$.

ϕ *is* $\mathsf{Ł}_\triangle$-valid *iff* $v(\phi)=1$ *for every* v. Γ entails χ *(denoted* $\Gamma\models_{\mathsf{Ł}_\triangle}\chi$*) iff for every* v *s.t.* $v(\phi)=1$ *for all* $\phi\in\Gamma$*, it holds that* $v(\chi)=1$ *as well.*

Remark 2. Note that $\triangle$, $\sim$, and $\rightarrow$ can be used to define all other connectives as follows.

$$\phi\vee\chi\coloneqq(\phi\rightarrow\chi)\rightarrow\chi \qquad \phi\wedge\chi\coloneqq\sim(\sim\phi\vee\sim\chi) \qquad \phi\oplus\chi\coloneqq\sim\phi\rightarrow\chi$$
$$\phi\odot\chi\coloneqq\sim(\phi\rightarrow\sim\chi) \qquad \phi\ominus\chi\coloneqq\phi\odot\sim\chi$$

To facilitate the presentation, we recall the Hilbert calculus for $\mathsf{Ł}_\triangle$. It can be obtained by adding $\triangle$ axioms and rules from [1], [24, Defenition 2.4.5], or [10, Chapter I,2.2.1] to the Hilbert-style calculus for $\mathsf{Ł}$ from [27, §6.2].

Definition 6 ($\mathcal{H}\mathsf{Ł}_\triangle$ — the Hilbert-style calculus for $\mathsf{Ł}_\triangle$). *The calculus contains the following axioms and rules.*

w: $\phi\rightarrow(\chi\rightarrow\phi)$.
sf: $(\phi\rightarrow\chi)\rightarrow((\chi\rightarrow\psi)\rightarrow(\phi\rightarrow\psi))$.
waj: $((\phi\rightarrow\chi)\rightarrow\chi)\rightarrow((\chi\rightarrow\phi)\rightarrow\phi)$.
co: $(\sim\chi\rightarrow\sim\phi)\rightarrow(\phi\rightarrow\chi)$.

MP: $\dfrac{\phi \quad \phi \to \chi}{\chi}$.
$\triangle$1: $\triangle\phi \vee {\sim}\triangle\phi$.
$\triangle$2: $\triangle\phi \to \phi$.
$\triangle$3: $\triangle\phi \to \triangle\triangle\phi$.
$\triangle$4: $\triangle(\phi \vee \chi) \to \triangle\phi \vee \triangle\chi$.
$\triangle$5: $\triangle(\phi \to \chi) \to \triangle\phi \to \triangle\chi$.
$\triangle$nec: $\dfrac{\phi}{\triangle\phi}$.

Łukasiewicz logic is known to lack compactness [24, Remark 3.2.14], whence, $\mathcal{H}\text{Ł}_\triangle$ is only *weakly complete*.

Proposition 1 (Weak completeness of $\mathcal{H}\text{Ł}_\triangle$). *Let $\Gamma \subseteq \mathcal{L}_{\text{Ł}}$ be finite. Then*

$$\Gamma \models_{\text{Ł}_\triangle} \phi \text{ iff } \Gamma \vdash_{\mathcal{H}\text{Ł}_\triangle} \phi$$

Definition 7 ($\text{Ł}^2_\triangle$). *The language is constructed using the following grammar.*

$$\mathcal{L}_{\text{Ł}^2_\triangle} \ni \phi \coloneqq p \in \texttt{Prop} \mid \neg\phi \mid {\sim}\phi \mid \triangle\phi \mid (\phi \to \phi)$$

The semantics is given by two *valuations v_1 (support of truth) and v_2 (support of falsity) $v_1, v_2 : \texttt{Prop} \to [0,1]$ that are extended as follows.*

$$\begin{aligned} v_1(\neg\phi) &= v_2(\phi) & v_2(\neg\phi) &= v_1(\phi)\\ v_1({\sim}\phi) &= {\sim}_{\text{Ł}} v_1(\phi) & v_2({\sim}\phi) &= {\sim}_{\text{Ł}} v_2(\phi)\\ v_1(\triangle\phi) &= \triangle_{\text{Ł}} v_1(\phi) & v_2(\triangle\phi) &= {\sim}_{\text{Ł}}\triangle_{\text{Ł}}{\sim}_{\text{Ł}} v_2(\phi)\\ v_1(\phi \to \chi) &= v_1(\phi) \to_{\text{Ł}} v_1(\chi) & v_2(\phi \to \chi) &= v_2(\chi) \ominus_{\text{Ł}} v_2(\phi) \end{aligned}$$

*We say that ϕ is $\text{Ł}^2_\triangle$-*valid *iff for every v_1 and v_2, it holds that $v_1(\phi) = 1$ and $v_2(\phi) = 0$.*

Remark 3. Again, the remaining connectives can be defined as in Remark 2. Furthermore, when there is no risk of confusion, we write $v(\phi) = (x, y)$ to designate that $v_1(\phi) = x$ and $v_2(\phi) = y$.

We are now ready to present the two-layered logics. We begin with $\mathsf{Pr}^{\text{Ł}^2}_\triangle$ from [7].

Definition 8 ($\mathsf{Pr}^{\text{Ł}^2}_\triangle$: language and semantics). *The language of $\mathsf{Pr}^{\text{Ł}^2}_\triangle$ is given by the following grammar*

$$\mathcal{L}_{\mathsf{Pr}^{\text{Ł}^2}_\triangle} \ni \alpha \coloneqq \mathsf{Pr}\phi \mid {\sim}\alpha \mid \neg\alpha \mid \triangle\alpha \mid (\alpha \to \alpha) \qquad (\phi \in \mathcal{L}_{\mathsf{BD}})$$

A $\mathsf{Pr}^{\text{Ł}^2}_\triangle$ model is a tuple $\mathbb{M} = \langle \mathfrak{M}, \mu, e_1, e_2 \rangle$ with $\langle \mathfrak{M}, \mu \rangle$ being a BD model with $\pm$-probability and $e_1, e_2 : \mathcal{L}_{\mathsf{Pr}^{\text{Ł}^2}_\triangle} \to [0,1]$ s.t. $e_1(\mathsf{Pr}\phi) = \mu(|\phi|^+)$, $e_2(\mathsf{Pr}\phi) = \mu(|\phi|^-)$, and the values of complex formulas being computed following Definition 7. We say that α is $\mathsf{Pr}^{\text{Ł}^2}_\triangle$ valid *iff $e(\alpha) = (1, 0)$ in every model.*

Definition 9 ($4\mathsf{Pr}^{\text{Ł}_\triangle}$: language and semantics). *The language of $4\mathsf{Pr}^{\text{Ł}_\triangle}$ is constructed by the following grammar:*

$$\mathcal{L}_{4\mathsf{Pr}^{\text{Ł}_\triangle}} \ni \alpha \coloneqq \mathsf{Bl}\phi \mid \mathsf{Db}\phi \mid \mathsf{Cf}\phi \mid \mathsf{Uc}\phi \mid {\sim}\alpha \mid \triangle\alpha \mid (\alpha \to \alpha) \qquad (\phi \in \mathcal{L}_{\mathsf{BD}})$$

A $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$ *model is a tuple* $\mathbb{M} = \langle \mathfrak{M}, \mu_{\mathbf{4}}, e\rangle$ *with* $\langle \mathfrak{M}, \mu_{\mathbf{4}}\rangle$ *being a* BD *model with* $\mathbf{4}$-*probability s.t.* $e(\mathsf{Bl}\phi) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{b}})$, $e(\mathsf{Db}\phi) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{d}})$, $e(\mathsf{Cf}\phi) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}})$, $e(\mathsf{Uc}\phi) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{u}})$, *and the values of complex formulas computed via Definition 5. We say that* α *is* $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$ valid *iff* $e(\alpha) = 1$ *in every model. A set of formulas* Γ entails α ($\Gamma \models_{\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}} \alpha$) *iff there is no* $\mathbb{M}$ *s.t.* $e(\gamma) = 1$ *for every* $\gamma \in \Gamma$ *but* $e(\alpha) \neq 1$.

Remark 4. Note that we are going to prove only the weak completeness. In addition, BD is a tabular logic, whence there exist only finitely many pairwise non-equivalent formulas over a finite set of variables. Thus, we do not need to explicitly assume that the underlying BD models are finite.

Convention 4 *We will further call formulas of the form* $\mathsf{X}\phi$ ($\phi \in \mathcal{L}_{\mathsf{BD}}$, $\mathsf{X} \in \{\mathsf{Pr}, \mathsf{Bl}, \mathsf{Db}, \mathsf{Cf}, \mathsf{Uc}\}$) modal atoms. *We interpret the value of a modal atom as a degree of certainty that the agent has in* ϕ. *For example,* $e(\mathsf{Pr}p) = (\frac{3}{4}, \frac{1}{2})$ *means that the agent's certainty in* p *is* $\frac{3}{4}$ *and in* $\neg p$ *is* $\frac{1}{2}$. *Similarly,* $e(\mathsf{Cf}q) = \frac{1}{3}$ *is construed as 'the agent is conflicted w.r.t.* q *to the degree* $\frac{1}{3}$*'.*

To make the semantics clearer, we provide the following example.

Example 1. Consider the following BD model.

$$w_0 : p^{\pm}, \not{q} \qquad\qquad w_1 : p^-, q^-$$

And let $\mu = \mu_{\mathbf{4}}$ be defined as follows: $\mu(\{w_0\}) = \frac{2}{3}$, $\mu(\{w_1\}) = \frac{1}{3}$, $\mu(W) = 1$, $\mu(\varnothing) = 0$. It is easy to check that μ satisfies the conditions of Definitions 2 and 3. Now let e be the $\text{Ł}^2_\triangle$ *valuation* and $e_{\mathbf{4}}$ the $\text{Ł}_\triangle$ *valuation* induced by μ and $\mu_{\mathbf{4}}$, respectively.

Consider two BD formulas: $p \vee q$ and p. We have $e(\mathsf{Pr}(p \vee q)) = \left(\frac{2}{3}, \frac{1}{3}\right)$ and $e(\mathsf{Pr}p) = \left(\frac{2}{3}, 1\right)$. In $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$, we have $e_{\mathbf{4}}(\mathsf{Bl}(p \vee q)) = \frac{2}{3}$, $e_{\mathbf{4}}(\mathsf{Db}(p \vee q)) = \frac{1}{3}$, $e_{\mathbf{4}}(\mathsf{Cf}p) = \frac{2}{3}$, $e_{\mathbf{4}}(\mathsf{Cf}(p \vee q)), e_{\mathbf{4}}(\mathsf{Uc}(p \vee q)) = 0$, $e_{\mathbf{4}}(\mathsf{Bl}p), e(\mathsf{Uc}p) = 0$, $e_{\mathbf{4}}(\mathsf{Cf}p) = \frac{2}{3}$, and $e(\mathsf{Db}p) = \frac{1}{3}$.

The following property of $\mathsf{Pr}^{\text{Ł}^2}_\triangle$ is going to be useful further in the section.

Lemma 1. *Let* $\alpha \in \mathcal{L}_{\mathsf{Pr}^{\text{Ł}^2}_\triangle}$. *Then,* α *is* $\mathsf{Pr}^{\text{Ł}^2}_\triangle$ *valid iff* $e_1(\alpha) = 1$ *in every* $\mathsf{Pr}^{\text{Ł}^2}_\triangle$ *model.*

Proof. Let $\mathbb{M} = \langle W, v^+, v^-, \mu, e_1, e_2\rangle$ be a $\mathsf{Pr}^{\text{Ł}^2}_\triangle$ model s.t. $e_2(\alpha) \neq 0$. We construct a model $\mathbb{M}^* = \langle W, (v^*)^+, (v^*)^-, \mu, e^*_1, e^*_2\rangle$ where $e^*_1(\alpha) \neq 1$. To do this, we define new BD valuations $(v^*)^+$ and $(v^*)^-$ on W as follows.

$$w \in v^+(p), w \notin v^-(p) \text{ then } w \in (v^*)^+(p), w \notin (v^*)^-(p)$$
$$w \in v^+(p), v^-(p) \text{ then } w \notin (v^*)^+(p), (v^*)^-(p)$$
$$w \notin v^+(p), v^-(p) \text{ then } w \in (v^*)^+(p), (v^*)^-(p)$$
$$w \notin v^+(p), w \in v^-(p) \text{ then } w \notin (v^*)^+(p), w \in (v^*)^-(p)$$

It can be easily checked by induction on $\phi \in \mathcal{L}_{\mathsf{BD}}$ that

$$|\phi|^+_{\mathbb{M}} = W \setminus |\phi|^-_{\mathbb{M}^*} \qquad\qquad |\phi|^-_{\mathbb{M}} = W \setminus |\phi|^+_{\mathbb{M}^*}$$

Now, since we can w.l.o.g. assume that μ is a (classical) probability measure on W (recall Remark 1), we have that

$$e^*(\mathsf{Pr}\phi) = (1 - \mu(|\phi|^-), 1 - \mu(|\phi|^+)) = (1 - e_2(\mathsf{Pr}\phi), 1 - e_1(\mathsf{Pr}\phi))$$

Observe that if $e(\alpha) = (x, y)$, then $e(\neg{\sim}\alpha) = (1 - y, 1 - x)$. Furthermore, it is straightforward to verify that the following formulas are valid.

$$\neg{\sim}\neg\alpha \leftrightarrow \neg\neg{\sim}\alpha \qquad \neg{\sim}{\sim}\alpha \leftrightarrow {\sim}\neg{\sim}\alpha$$
$$\neg{\sim}\triangle\alpha \leftrightarrow \triangle\neg{\sim}\alpha \qquad \neg{\sim}(\alpha \to \alpha') \leftrightarrow \neg{\sim}\alpha \to \neg{\sim}\alpha'$$

Hence, $e^*(\alpha) = (1 - e_2(\alpha), 1 - e_1(\alpha))$ for every $\alpha \in \mathcal{L}_{\mathsf{Pr}^{\text{Ł}^2}_{\triangle}}$. The result follows.

At first glance, $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$ gives a more fine-grained view on a BD model than $\mathsf{Pr}^{\text{Ł}^2}_{\triangle}$ since it can evaluate each extension of a given $\phi \in \mathcal{L}_{\mathsf{BD}}$, while $\mathsf{Pr}^{\text{Ł}^2}_{\triangle}$ always considers $|\phi|^+$ and $|\phi|^-$ together. In the remainder of the section, we show that the two logics have, in fact, the same expressivity.

One can see from Definition 8 that $\neg\mathsf{Pr}\phi \leftrightarrow \mathsf{Pr}\neg\phi$. Furthermore, Ł^2 admits $\neg$ negation normal forms and is a conservative extension of Ł [8,5]. Thus, it is possible to push all $\neg$'s occurring in $\alpha \in \mathcal{L}_{\mathsf{Pr}^{\text{Ł}^2}_{\triangle}}$ to modal atoms. We will use this fact to establish the embeddings of $\mathsf{Pr}^{\text{Ł}^2}_{\triangle}$ and $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$ into one another.

Definition 10. *Let $\alpha \in \mathcal{L}_{\mathsf{Pr}^{\text{Ł}^2}_{\triangle}}$. $\alpha^\neg$ is produced from α by successively applying the following transformations.*

$$\neg\mathsf{Pr}\phi \rightsquigarrow \mathsf{Pr}\neg\phi \qquad \neg\neg\alpha \rightsquigarrow \alpha \qquad \neg{\sim}\alpha \rightsquigarrow {\sim}\neg\alpha$$
$$\neg(\alpha \to \alpha') \rightsquigarrow {\sim}(\neg\alpha' \to \neg\alpha) \qquad \neg\triangle\alpha \rightsquigarrow {\sim}\triangle{\sim}\neg\alpha$$

It is easy to check that $e(\alpha) = e(\alpha^\neg)$ in every $\mathsf{Pr}^{\text{Ł}^2}_{\triangle}$ model.

Definition 11. *Let $\alpha \in \mathcal{L}_{\mathsf{Pr}^{\text{Ł}^2}_{\triangle}}$ be $\neg$-free, we define $\alpha^{\mathbf{4}} \in \mathcal{L}_{\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}}$ as follows.*

$$(\mathsf{Pr}\phi)^{\mathbf{4}} = \mathsf{Bl}\phi \oplus \mathsf{Cf}\phi$$
$$(\heartsuit\alpha)^{\mathbf{4}} = \heartsuit\alpha^{\mathbf{4}} \qquad (\heartsuit \in \{\triangle, {\sim}\})$$
$$(\alpha \to \alpha')^{\mathbf{4}} = \alpha^{\mathbf{4}} \to \alpha'^{\mathbf{4}}$$

Let $\beta \in \mathcal{L}_{\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}}$. We define $\beta^\pm$ as follows.

$$(\mathsf{Bl}\phi)^\pm = \mathsf{Pr}\phi \ominus \mathsf{Pr}(\phi \wedge \neg\phi)$$
$$(\mathsf{Cf}\phi)^\pm = \mathsf{Pr}(\phi \wedge \neg\phi)$$
$$(\mathsf{Uc}\phi)^\pm = {\sim}\mathsf{Pr}(\phi \vee \neg\phi)$$
$$(\mathsf{Db}\phi)^\pm = \mathsf{Pr}\neg\phi \ominus \mathsf{Pr}(\phi \wedge \neg\phi)$$
$$(\heartsuit\beta)^\pm = \heartsuit\beta^\pm \qquad (\heartsuit \in \{\triangle, {\sim}\})$$
$$(\beta \to \beta')^\pm = \beta^\pm \to \beta'^\pm$$

Theorem 1. *$\alpha \in \mathcal{L}_{\mathsf{Pr}^{\text{Ł}^2}_{\triangle}}$ is $\mathsf{Pr}^{\text{Ł}^2}_{\triangle}$ valid iff $(\alpha^\neg)^{\mathbf{4}}$ is $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$ valid.*

Proof. Let w.l.o.g. $\mathbb{M} = \langle W, v^+, v^-, \mu, e_1, e_2\rangle$ be a BD model with $\pm$-probability where μ is a *classical* probability measure and let $e(\alpha) = (x, y)$. We show that in the BD model $\mathbb{M}_{\mathbf{4}} = \langle W, v^+, v^-, \mu, e_1\rangle$ with *four-probability* μ, $e_1((\alpha^\neg)^{\mathbf{4}}) = x$. This is sufficient to prove the result. Indeed, by Lemma 1, it suffices to verify that $e_1(\alpha) = 1$ for every e_1, to establish the validity of $\alpha \in \mathcal{L}_{\mathsf{Pr}^{\text{Ł}^2}_{\triangle}}$.

We proceed by induction on $\alpha^\neg$ (recall that $\alpha \leftrightarrow \alpha^\neg$ is $\mathsf{Pr}^{\text{Ł}^2}_{\triangle}$ valid). If $\alpha = \mathsf{Pr}\phi$, then $e_1(\mathsf{Pr}\phi) = \mu(|\phi|^+) = \mu(|\phi|^{\mathsf{b}} \cup |\phi|^{\mathsf{c}})$. But $|\phi|^{\mathsf{b}}$ and $|\phi|^{\mathsf{c}}$ are disjoint, whence

$\mu(|\phi|^{\mathsf{b}} \cup |\phi|^{\mathsf{c}}) = \mu(|\phi|^{\mathsf{b}}) + \mu(|\phi|^{\mathsf{c}})$, and since $\mu(|\phi|^{\mathsf{b}}) + \mu(|\phi|^{\mathsf{c}}) \leq 1$, we have that $e_1(\mathsf{Bl}\phi \oplus \mathsf{Cf}\phi) = \mu(|\phi|^{\mathsf{b}}) + \mu(|\phi|^{\mathsf{c}}) = e_1(\mathsf{Pr}\phi)$, as required.

The induction steps are straightforward since the semantic conditions of support of truth in $\text{Ł}^2_\triangle$ coincide with the semantics of $\text{Ł}_\triangle$ (cf. Definitions 7 and 5).

Theorem 2. *$\beta \in \mathcal{L}_{\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}}$ is $\mathcal{L}_{\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}}$ valid iff $\beta^\pm$ is $\mathsf{Pr}^{\text{Ł}^2}_\triangle$ valid.*

Proof. Assume w.l.o.g. that $\mathbb{M} = \langle W, v^+, v^-, \mu_{\mathbf{4}}, e\rangle$ is a BD model with a **4**-probability where $\mu_{\mathbf{4}}$ is a classical probability measure and $e(\beta) = x$. We define a BD model with $\pm$-probability $\mathbb{M}^\pm = \langle W, v^+, v^-, \mu_{\mathbf{4}}, e_1, e_2\rangle$ and show that $e_1(\beta^\pm) = x$. Again, it is sufficient for us by Lemma 1.

We proceed by induction on β. If $\beta = \mathsf{Bl}\phi$, then $e(\mathsf{Bl}\phi) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{b}})$. Now observe that $\mu_{\mathbf{4}}(|\phi|^+) = \mu(|\phi|^{\mathsf{b}} \cup |\phi|^{\mathsf{c}}) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}})$ since $|\phi|^{\mathsf{b}}$ and $|\phi|^{\mathsf{c}}$ are disjoint. But $\mu_{\mathbf{4}}(|\phi|^+) = e_1(\mathsf{Pr}\phi)$ and $\mu_{\mathbf{4}}(|\phi|^{\mathsf{c}}) = \mu_{\mathbf{4}}(|\phi \wedge \neg\phi|^+)$ since $|\phi \wedge \neg\phi|^+ = |\phi|^{\mathsf{c}}$. Thus, $\mu_{\mathbf{4}}(|\phi|^{\mathsf{b}}) = e_1(\mathsf{Pr}\phi \ominus \mathsf{Pr}(\phi \wedge \neg\phi))$ as required.

Other basis cases of $\mathsf{Cf}\phi$, $\mathsf{Uc}\phi$, and $\mathsf{Db}\phi$ can be tackled in a similar manner. The induction steps are straightforward since the support of truth in $\text{Ł}^2_\triangle$ coincides with semantical conditions in $\text{Ł}_\triangle$.

Remark 5. Theorem 1 and 2 mean, in a sense, that $\mathsf{Pr}^{\text{Ł}^2}_\triangle$ and $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$ can be treated as syntactic variants of one another. Conceptually, however, they are somewhat different. Namely, $\mathsf{Pr}^{\text{Ł}^2}_\triangle$ assigns *two independent measures* to each formula ϕ corresponding to the likelihoods of ϕ itself and $\neg\phi$. On the other hand, $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$ treats the extensions ϕ as a separation of the underlying sample set into four parts whose measures must add up to 1.

4 Hilbert-style axiomatisation of $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$

Let us proceed to the axiomatisation of $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$. Since its outer layer expands $\text{Ł}_\triangle$, we will need to encode the conditions on $\mu_{\mathbf{4}}$ therein. Furthermore, since Ł (and hence, $\text{Ł}_\triangle$) is not compact [24, Remark 3.2.14], our axiomatisation can only be *weakly complete* (i.e., complete w.r.t. finite theories).

The axiomatisation will consist of two types of axioms: those that axiomatise $\text{Ł}_\triangle$ and modal axioms that encode the conditions from Definition 3. For the sake of brevity, we will compress the axiomatisation of $\text{Ł}_\triangle$ into one axiom that allows us to use $\text{Ł}_\triangle$ theorems[5] without proof.

Definition 12 ($\mathcal{H}\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$ — Hilbert-style calculus for $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$). *The calculus $\mathcal{H}\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$ consists of the following axioms and rules.*

$\text{Ł}_\triangle$: $\text{Ł}_\triangle$ *valid formulas instantiated in* $\mathcal{L}_{\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}}$.

equiv: $\mathsf{X}\phi \leftrightarrow \mathsf{X}\chi$ *for every* $\phi, \chi \in \mathcal{L}_{\mathsf{BD}}$ *s.t.* $\phi \dashv\vdash \chi$ *is* BD*-valid and* $\mathsf{X} \in \{\mathsf{Bl}, \mathsf{Db}, \mathsf{Cf}, \mathsf{Uc}\}$.

contr: $\sim\mathsf{Bl}(\phi \wedge \neg\phi)$*;* $\mathsf{Cf}\phi \leftrightarrow \mathsf{Cf}(\phi \wedge \neg\phi)$.

[5] A Hilbert-style calculus for Ł can be found in, e.g. [27], and the axioms for $\triangle$ in [1]. A concise presentation of a Hilbert-style calculus for $\text{Ł}_\triangle$ is also given in [7].

neg: $\mathsf{Bl}\neg\phi \leftrightarrow \mathsf{Db}\phi$; $\mathsf{Cf}\neg\phi \leftrightarrow \mathsf{Cf}\phi$.

mon: $(\mathsf{Bl}\phi \oplus \mathsf{Cf}\phi) \rightarrow (\mathsf{Bl}\chi \oplus \mathsf{Cf}\chi)$ *for every* $\phi, \chi \in \mathcal{L}_{\mathsf{BD}}$ *s.t.* $\phi \vdash \chi$ *is* BD*-valid.*

part1: $\mathsf{Bl}\phi \oplus \mathsf{Db}\phi \oplus \mathsf{Cf}\phi \oplus \mathsf{Uc}\phi$.

part2: $((\mathsf{X}_1\phi \oplus \mathsf{X}_2\phi \oplus \mathsf{X}_3\phi \oplus \mathsf{X}_4\phi) \ominus \mathsf{X}_4\phi) \leftrightarrow (\mathsf{X}_1\phi \oplus \mathsf{X}_2\phi \oplus \mathsf{X}_3\phi)$ *with* $\mathsf{X}_i \neq \mathsf{X}_j$, $\mathsf{X}_i \in \{\mathsf{Bl}, \mathsf{Db}, \mathsf{Cf}, \mathsf{Uc}\}$.

ex: $(\mathsf{Bl}(\phi\vee\chi)\oplus\mathsf{Cf}(\phi\vee\chi)) \leftrightarrow ((\mathsf{Bl}\phi\oplus\mathsf{Cf}\phi)\ominus(\mathsf{Bl}(\phi\wedge\chi)\oplus\mathsf{Cf}(\phi\wedge\chi))\oplus(\mathsf{Bl}\chi\oplus\mathsf{Cf}\chi))$.

MP: $\dfrac{\alpha \quad \alpha \rightarrow \alpha'}{\alpha'}$.

$\triangle$nec: $\dfrac{\mathcal{H}\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle} \vdash \alpha}{\mathcal{H}\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle} \vdash \triangle\alpha}$.

The axioms above are simple translations of properties from Definition 3. We split **part** in two axioms to ensure that the values of $\mathsf{Bl}\phi$, $\mathsf{Db}\phi$, $\mathsf{Cf}\phi$, and $\mathsf{Uc}\phi$ sum up exactly to 1.

Theorem 3. *Let* $\Xi \subseteq \mathcal{L}_{\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}}$ *be finite. Then* $\Xi \models_{\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}} \alpha$ *iff* $\Xi \vdash_{\mathcal{H}\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}} \alpha$.

Proof. Soundness can be established by the routine check of the axioms' validity. Thus, we prove completeness. We reason by contraposition. Assume that $\Xi \nvdash_{\mathcal{H}\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}} \alpha$. Now, observe that $\mathcal{H}\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ proofs are, actually, $\mathcal{H}\mathsf{Ł}_\triangle$ proofs with additional probabilistic axioms. Let Ξ^* stand for Ξ extended with probabilistic axioms built over all pairwise non-equivalent $\mathcal{L}_{\mathsf{BD}}$ formulas constructed from $\mathtt{Prop}[\Xi \cup \{\alpha\}]$. Clearly, $\Xi^* \nvdash_{\mathcal{H}\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}} \alpha$ either. Moreover, Ξ^* is finite as well since BD is tabular (and whence, there exist only finitely many pairwise non-equivalent $\mathcal{L}_{\mathsf{BD}}$ formulas over a finite set of variables). Now, by the weak completeness of $\mathcal{H}\mathsf{Ł}_\triangle$ (Proposition 1), there exists an $\mathsf{Ł}_\triangle$ valuation e on $[0,1]$ s.t. $e[\Xi^*] = 1$ and $e(\alpha) \neq 1$.

It remains to construct a $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ model $\mathbb{M}$ falsifying $\Xi^* \models_{\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}} \alpha$ using e. We proceed as follows. First, we set $W = 2^{\mathsf{Lit}[\Xi^* \cup \{\alpha\}]}$, and for every $w \in W$ define $w \in v^+(p)$ iff $p \in w$ and $w \in v^-(p)$ iff $\neg p \in w$. We extend the valuations to $\phi \in \mathcal{L}_{\mathsf{BD}}$ in the usual manner. Then, for $\mathsf{X}\phi \in \mathtt{Sf}[\Xi^* \cup \{\alpha\}]$, we set $\mu_{\mathbf{4}}(|\phi|^{\mathsf{x}}) = e(\mathsf{X}\phi)$ according to modality X.

It remains to extend $\mu_{\mathbf{4}}$ to the whole 2^W. Observe, however, that any map from 2^W to $[0,1]$ that extends $\mu_{\mathbf{4}}$ is, in fact, a $\mathbf{4}$-probability. Indeed, all requirements from Definition 3 concern *only the extensions of formulas*. But the model is finite, BD is tabular, and Ξ^* contains all the necessary instances of probabilistic axioms and $e[\Xi^*] = 1$, whence all constraints on the formulas are satisfied.

Remark 6. Observe that we could use a *classical probability measure* in the proof of Theorem 3 because of [26, Theorem 5]. This, however, would require us to show that the extensions of formulas form a subalgebra of 2^W. On the other hand, it is simpler to use $\mathbf{4}$-probabilities instead of classical probabilities since we can immediately extend them to the full powerset from the extensions of formulas by Definition 3.

5 Decidability and complexity

In the completeness proof, we reduced $\mathcal{H}\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$ proofs to $Ł_\triangle$ proofs. We know that validity and finitary entailment of $Ł_\triangle$ are coNP-complete (since $Ł$ is coNP-complete and $\triangle$ has truth-functional semantics).

Likewise, $\mathsf{Pr}^{Ł^2}_\triangle$ proofs are also reducible to $Ł^2$ proofs (cf. [7, Theorem 4.24]) from substitution instances of axioms $\mathsf{Pr}\phi \rightarrow \mathsf{Pr}\chi$ (for $\phi \models_{\mathsf{BD}} \chi$), $\neg\mathsf{Pr}\phi \leftrightarrow \mathsf{Pr}\neg\phi$, and $\mathsf{Pr}(\phi \vee \chi) \leftrightarrow (\mathsf{Pr}\phi \ominus \mathsf{Pr}(\phi \wedge \chi)) \oplus \mathsf{Pr}\chi$. Thus, it is clear that the validity and satisfiability of $\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$ and $\mathsf{Pr}^{Ł^2}_\triangle$ are coNP-hard and NP-hard, respectively.

In this section, we provide a simple decision procedure for $\mathsf{Pr}^{Ł^2}_\triangle$ and $\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$ and show that their satisfiability and validity are NP- and coNP-complete, respectively. Namely, we adapt constraint tableaux for $Ł^2$ defined in [5] and expand them with rules for $\triangle$. We then adapt the NP-completeness proof $\mathsf{FP}(Ł)$ from [25] to establish our result.

Definition 13 (Constraint tableaux for $Ł^2_\triangle$ — $\mathcal{T}\left(Ł^2_\triangle\right)$). *Branches contain* labelled formulas *of the form $\phi \leqslant_1 i$, $\phi \leqslant_2 i$, $\phi \geqslant_1 i$, or $\phi \geqslant_2 i$, and* numerical constraints *of the form $i \le j$ with $i, j \in [0,1]$.*

Each branch can be extended by an application of one of the rules below.

$$\neg\leqslant_1 \frac{\neg\phi \leqslant_1 i}{\phi \leqslant_2 i} \quad \neg\leqslant_2 \frac{\neg\phi \leqslant_2 i}{\phi \leqslant_1 i} \quad \neg\geqslant_1 \frac{\neg\phi \geqslant_1 i}{\phi \geqslant_2 i} \quad \neg\geqslant_2 \frac{\neg\phi \geqslant_2 i}{\phi \geqslant_1 i}$$

$$\sim\leqslant_1 \frac{\sim\phi \leqslant_1 i}{\phi \geqslant_1 1-i} \quad \sim\leqslant_2 \frac{\sim\phi \leqslant_2 i}{\phi \geqslant_2 1-i} \quad \sim\geqslant_1 \frac{\sim\phi \geqslant_1 i}{\phi \leqslant_1 1-i} \quad \sim\geqslant_2 \frac{\sim\phi \geqslant_2 i}{\phi \leqslant_2 1-i}$$

$$\triangle\geqslant_1 \frac{\triangle\phi \geqslant_1 i}{i \le 0 \left| \begin{array}{c} \phi \geqslant_1 j \\ j \ge 1 \end{array} \right.} \quad \triangle\leqslant_1 \frac{\triangle\phi \leqslant_1 i}{i \ge 1 \left| \begin{array}{c} \phi \leqslant_1 j \\ j < 1 \end{array} \right.} \quad \triangle\leqslant_2 \frac{\triangle\phi \leqslant_2 i}{i \ge 1 \left| \begin{array}{c} \phi \leqslant j \\ j \le 0 \end{array} \right.} \quad \triangle\geqslant_2 \frac{\triangle\phi \geqslant_2 i}{i \le 0 \left| \begin{array}{c} \phi \geqslant j \\ j > 0 \end{array} \right.}$$

$$\rightarrow\leqslant_1 \frac{\phi_1 \rightarrow \phi_2 \leqslant_1 i}{i \ge 1 \left| \begin{array}{c} \phi_1 \geqslant_1 1-i+j \\ \phi_2 \leqslant_1 j \\ j \le i \end{array} \right.} \qquad \rightarrow\leqslant_2 \frac{\phi_1 \rightarrow \phi_2 \leqslant_2 i}{\begin{array}{c} \phi_1 \geqslant_2 j \\ \phi_2 \leqslant_2 i+j \end{array}}$$

$$\rightarrow\geqslant_1 \frac{\phi_1 \rightarrow \phi_2 \geqslant_1 i}{\begin{array}{c} \phi_1 \leqslant_1 1-i+j \\ \phi_2 \geqslant_1 j \end{array}} \qquad \rightarrow\geqslant_2 \frac{\phi_1 \rightarrow \phi_2 \geqslant_2 i}{i \le 0 \left| \begin{array}{c} \phi_1 \leqslant_2 j \\ \phi_2 \geqslant_2 i+j \\ j \le 1-i \end{array} \right.}$$

Let i's be in $[0,1]$ and x's be variables ranging over the real interval $[0,1]$. We define the translation τ from labelled formulas to linear inequalities as follows:

$$\tau(\phi \leqslant_1 i) = x^L_\phi \le i;\ \tau(\phi \geqslant_1 i) = x^L_\phi \ge i;\ \tau(\phi \leqslant_2 i) = x^R_\phi \le i;\ \tau(\phi \geqslant_2 i) = x^R_\phi \ge i$$

Let $\bullet \in \{\leqslant_1, \geqslant_1\}$ and $\circ \in \{\leqslant_2, \geqslant_2\}$. A tableau branch

$$\mathcal{B} = \{\phi_1 \circ i_1, \ldots, \phi_m \circ i_m, \phi'_1 \bullet j_1, \ldots, \phi'_n \bullet j_n, k_1 \le l_1, \ldots, k_q \le l_q\}$$

is closed *if the system of inequalities*

$$\tau(\phi_1 \circ i_1), \ldots, \tau(\phi_m \circ i_m), \tau(\phi'_1 \bullet j_1), \ldots, \tau(\phi'_n \bullet j_n), k_1 \le l_1, \ldots, k_q \le l_q$$

does not have solutions. Otherwise, $\mathcal{B}$ *is* open. *A tableau is* closed *if all its branches are closed.* ϕ *has a* $\mathcal{T}\left(\mathsf{Ł}^2_\triangle\right)$ proof *if the tableau beginning with* $\{\phi \leqslant_1 c, c < 1\}$ *is closed.*

Observe that the $\rightarrow$ and $\sim$ rules for $\leqslant_1$ coincide with the analoguous rules in the constraint tableaux for $\mathsf{Ł}$ as given in [20,21,23]. Thus, we can use the calculus both for $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ and $\mathsf{Pr}^{\mathsf{Ł}^2}_\triangle$. Note also that we need to build only one tableau for $\mathcal{L}_{\mathsf{Pr}^{\mathsf{Ł}^2}_\triangle}$ formulas because of Lemma 1.

The next statement can be proved in the same manner as [5, Theorem 1].

Theorem 4 (Completeness of tableaux).
1. ϕ *is* $\mathsf{Ł}_\triangle$ *valid iff it has a* $\mathcal{T}\left(\mathsf{Ł}^2_\triangle\right)$ *proof.*
2. ϕ *is* $\mathsf{Ł}^2_\triangle$ *valid iff it has a* $\mathcal{T}\left(\mathsf{Ł}^2_\triangle\right)$ *proof.*

As we have already mentioned in the beginning of the section, the proof of NP-completeness is an adaptation of a similar proof from [25]. This, in turn, uses the reduction of Łukasiewicz formulas to bounded mixed-integer problems (bMIPs) as given in [20,21,22]. To make the paper self-contained, we state the required definitions and results here.

Definition 14 (Mixed-integer problem). *Let* $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ *and* $\mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{Z}^m$ *be variables, A and B be integer matrices and h an integer vector, and* $f(\mathbf{x}, \mathbf{y})$ *be a* $k+m$*-place linear function.*
1. A general MIP *is to find* $\mathbf{x}$ *and* $\mathbf{y}$ *s.t.* $f(\mathbf{x}, \mathbf{y}) = \min\{f(\mathbf{x}, \mathbf{y}) : A\mathbf{x}+B\mathbf{y} \geq h\}$*.*
2. *In* a bounded MIP (bMIP)*, all solutions should belong to* $[0, 1]$*.*

Proposition 2. *Bounded MIP is* NP*-complete.*

Theorem 5. *Satisfiability of* $\mathsf{Pr}^{\mathsf{Ł}^2}_\triangle$ *and* $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ *is* NP*-complete.*

Proof. Recall that $\mathsf{Pr}^{\mathsf{Ł}^2}_\triangle$ and $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ can be linearly embedded into one another (Theorems 1 and 2). Thus, it remains to provide a non-deterministic polynomial algorithm for one of these logics. We choose $\mathsf{Pr}^{\mathsf{Ł}^2}_\triangle$ since it has only one modality.

Let $\alpha \in \mathcal{L}_{\mathsf{Pr}^{\mathsf{Ł}^2}_\triangle}$. We can w.l.o.g. assume that $\neg$ occurs only in modal atoms and that in every modal atom $\mathsf{Pr}\phi_i$, ϕ_i is in negation normal form. Define α^* to be the result of the substitution of every $\neg p$ occurring in α with a new variable p^*. It is easy to check that α is satisfiable iff α^* is. We construct a satisfying valuation for α^*.

First, we replace every modal atom $\mathsf{Pr}\phi_i$ with a fresh variable q_{ϕ_i}. Denote the new formula $(\alpha^*)^-$. It is clear that the size of $(\alpha^*)^-$ ($|(\alpha^*)^-|$) is only linearly greater than $|\alpha|$. We construct a tableau beginning with $\{(\alpha^*)^- \geqslant_1 c, c \geq 1\}$. Every branch gives us an instance of a bMIP equivalent to the $\mathsf{Ł}$-satisfiability of $(\alpha^*)^-$: $(\alpha^*)^-$ is satisfiable iff at least one instance of a bMIP has a solution.

Now, write z_i for the values of q_{ϕ_i}'s in $(\alpha^*)^-$. Our instance of a bMIP also has additional variables x_j ranging over $[0, 1]$ as well as equalities $k = 1$ and $k' = 0$ obtained from entries $k \geq 1$ and $k' \leq 0$. It is clear that both the number of (in)equalities l_1 and the number of variables l_2 in the MIP are linear w.r.t. $|(\alpha^*)^-|$. Denote this instance MIP(1).

We need to show that z_i's are coherent as probabilities of ϕ_i's (here, $i \leq n$ indexes the modal atoms of $(\alpha^*)^-$). We introduce 2^n variables u_v indexed by n-letter words over $\{0,1\}$ and denoting whether the variables of ϕ_i's are true under v^+.[6] We let $a_{i,v} = 1$ when ϕ_i is true under v^+ and $a_{i,v} = 0$ otherwise. Now add new equalities denoted with MIP(2 exp) to MIP(1), namely, $\sum_v u_v = 1$ and $\sum_v (a_{i,v} \cdot u_v) = z_i$. It is clear that the new MIP (MIP(1)∪MIP(2 exp)) has a non-negative solution iff its corresponding branch is open. Furthermore, although there are l_2+2^n+n variables in MIP(1)∪MIP(2 exp), it has no more than l_1+n+1 (in)equalities. Thus by [18, Lemma 2.5], it has a non-negative solution with at most l_1+n+1 non-zero entries. We guess a list L of at most l_1+n+1 words v (its size is $n \cdot (l_1+n+1)$). We can now compute the values of $a_{i,v}$'s for $i \leq n$ and $v \in L$ and obtain a new MIP which we denote MIP(2poly): $\sum_{v \in L} u_v = 1$ and $\sum_{v \in L}(a_{i,v} \cdot u_v) = z_i$. It is clear that MIP(1) ∪ MIP(2poly) is of polynomial size w.r.t. $|\alpha|$ and has a non-negative solution iff the corresponding branch of the tableau is open. Thus, we can solve it in non-deterministic polynomial time as required.

6 Conclusion

We presented logic $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ formalising four-valued probabilities proposed in [26] using a two-layered expansion of Łukasiewicz logic with $\triangle$. We established faithful embeddings between $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ and $\mathsf{Pr}^{\mathsf{Ł}^2}_{\triangle}$, the logic of ±-probabilities [7]. Moreover, we constructed a sound and complete axiomatisation of $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ and proved its decidability using constraint tableaux for $\mathsf{Ł}_\triangle$.

Several questions remain open. In [7], we presented two-layered logics for reasoning with belief and plausibility functions. These logics employ a 'two-valued' interpretation of belief and plausibility (i.e., ϕ has two belief assignments: for ϕ and for $\neg\phi$). It would be instructive to axiomatise 'four-valued' belief and plausibility functions and formalise reasoning with those via a two-layered logic.

Moreover, we have been considering logics whose inner layer lacks implication. It is, however, reasonable to assume that an agent can assign certainty to conditional statements. Furthermore, there are expansions of BD with truth-functional implications (cf. [28] for examples). A natural next step now is to axiomatise paraconsistent probabilities defined over a logic with an implication.

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[6] Note that $\neg$ does not occur in $(\alpha^*)^-$ and thus we care only about e_1 and v^+. Furthermore, while n is the number of ϕ_i's, we can add superfluous modal atoms or variables to make it also the number of variables.

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