An Axiom System for Basic Hybrid Logic with Propositional Quantifiers

Patrick Blackburn[1], Torben Braüner[2], and Julie Lundbak Kofod[1]

[1] Department of Philosophy and Science Studies, Roskilde University
[2] Department of People and Technology, Roskilde University

Abstract. We present an axiom system for basic hybrid logic extended with propositional quantifiers (a second-order extension of basic hybrid logic) and prove its (basic and pure) strong completeness with respect to general models.

1 Introduction

We present an axiom system for basic hybrid logic augmented with propositional quantifiers — a second-order extension of basic hybrid logic — and prove its basic and pure completeness with respect to general models. A notable feature of our axiom system is that the universal instantiation rule for propositional quantification is restricted: variables can only be replaced by formulas that (a) are quantifier-free and (b) don't contain nominals in formula position.

Although this is primarily a technical paper, its roots are philosophical: it is part of an ongoing re-examination of the later work of Arthur Prior, a philosophical logician who is probably best known as the inventor of tense logic (see [10,15]). However Prior was also the founder of hybrid logic (see [8,4]) and he sometimes used propositional quantifiers to define what we now call nominals; these developments led Prior, shortly before his death in 1969, to explore such ideas as "quasi-modalities" and "egocentric logic".[3] We believe that the combination of contemporary hybrid logic and propositional quantification explored in this paper is a promising setting for better understanding Prior's later work.

We proceed as follows. In Section 2 we define the syntax of our languages, drawing special attention to what are called soft-QF formulas and soft-QF substitutions. In Section 3 we introduce general frames and models, give a Henkin-style satisfaction definition, and note some basic semantic lemmas. In Section 4 we define an axiom system and prove it sound, and then in Section 5 we prove its (basic and pure) strong completeness. Section 6 concludes and briefly discusses the links with Prior's later work.

[3] See, in particular, the technical papers in the new edition of his book *Papers on Time and Tense* [17], and the posthumous volume *Worlds, Times and Selves* [20].

2 Syntax and substitution

In this section we define basic hybrid logic with quantification over propositional variables, soft-QF formulas, and soft-QF substitutions. Basic hybrid logic is obtained by adding *nominals* and *satisfaction operators* to basic (propositional) modal logic. Nominals are usually written i, j, k; they are atomic symbols true at a unique world in any model. Nominals play two distinct syntactic roles. First, they can be used as atomic formulas, in exactly the same way as ordinary propositional variables p, q, and r can; because of the "true at a unique world" restriction on their interpretation, in this first role nominals can be thought of as 0-place world-naming modalities. Second, any nominal i can occur as a subscript to the symbol @. Any such $@_i$ is called a *satisfaction operator*, and for any formula φ, $@_i\varphi$ is true at a world w iff φ is true at the world that i names. Thus satisfaction operators are 1-place rigidifying modalities: they transform any proposition φ into a *rigid* proposition $@_i\varphi$, one that is either true at all worlds, or false at all worlds (depending on whether φ is true or false at the world that nominal i names). Any formula of the form $@_i\varphi$ is called a *satisfaction statement*.

We can now define what we mean by a basic hybrid language with propositional quantification. Let $PLET = \{c, b, a, \dots\}$ be a set of *propositional letters*, let $PVAR = \{p, q, r, \dots\}$ be a set of propositional variables, and let $NOM = \{i, j, k \dots\}$ be a set of *nominals*. We assume that all three sets are countable and pairwise disjoint, write $ATOM$ for $PLET \cup PVAR \cup NOM$ and call any element of $ATOM$ an *atomic symbol*. The basic hybrid language with propositional quantification $\mathcal{L}_{BHPQ}$ is built over $ATOM$ using the following grammar:

$$\varphi \ ::= \ c \mid p \mid i \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid @_i\varphi \mid \forall p\varphi,$$

where $c \in PLET$, $p \in PVAR$, and $i \in NOM$. Other booleans are defined as expected, and $\Diamond$ and $\exists$ are defined by $\neg\Box\neg \equiv \Diamond$ and $\neg\forall p\neg \equiv \exists p$.

It is clear from this definition that nominals can occur in formulas in two ways: either as an atomic formula (that is: in *formula position*) or as part of a satisfaction operator (that is: in *operator position*). Similarly, any propositional variable p can occur in a formula in two ways: either as an atomic formula (that is: in *formula position*) or right after the symbol $\forall$ (that is: in *binding position*). But unlike occurrences of $@_i$ (which do not bind occurrences of i), occurrences of $\forall p$ bind all the (free) occurrences of p they have scope over. Propositional letters, on the other hand, cannot be bound; they occur only in formula position. So propositional letters are essentially "propositional constants" and we will use them in our Lindenbaum Lemma as Henkin-style witnesses for existential quantifiers. We define free and bound propositional variables in the standard way, and write $FV(\varphi)$ for the set of free propositional variables in formula φ. A *sentence* is a formula that contains no free propositional variables.

The result of substituting a formula ψ for a propositional variable q occurring in some formula φ, written $\varphi[\psi/q]$, is defined in the expected way. It is always possible to carry out a substitution safely (that is: without accidental binding) by relabelling the bound variables in φ.

In the Hilbert-style system presented in Section 4, the universal instantiation axiom has a side condition: only *quantifer free formulas with no nominals in formula position* can be used to instantiate universal quantifications. We call such formulas soft-QF formulas. That is, soft-QF formulas are built using the grammar

$$\varphi \ ::= \ c \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid @_i\varphi,$$

where $c \in PLET$, $p \in PVAR$, and $i \in NOM$. A substitution $\varphi[\psi/q]$ is called a *soft-QF substitution* iff ψ is a soft-QF formula.

3 Semantics

We interpret $\mathcal{L}_{BHPQ}$ using a Henkin-style *general semantics*. That is, we shall use *general frames* and *general models* as our basic semantic structures: such structures restrict the domain over which the propositional quantifiers range. Writing $\mathcal{P}(W)$ for the powerset of the W, we define:

Definition 1 (Kripke frames, general frames, standard frames). *A* Kripke frame *is a pair $\mathcal{F} = \langle W, R\rangle$ where W is a non-empty set (of* worlds*) and R is a binary relation on W (the* accessibility relation *between worlds). A* general frame *is a pair $\mathcal{G} = \langle \mathcal{F}, \Pi\rangle$ where $\mathcal{F}$ is a Kripke frame and Π is a non-empty subset of $\mathcal{P}(W)$ that is closed under the following three conditions:*

- *relative complement: if $X \in \Pi$, then $W - X \in \Pi$*
- *intersection: if $X, Y \in \Pi$, then $X \cap Y \in \Pi$*
- *modal projection: if $X \in \Pi$ then $\{w \in W \mid \forall v(wRv \to v \in X)\} \in \Pi$.*

We call a subset Π of $\mathcal{P}(W)$ that satisfies these conditions a closed set of admissible propositions*, and we call the elements of Π* admissible propositions*. A general frame is called a* standard frame *iff $\Pi = \mathcal{P}(W)$.*

We are interested in general frames rather than standard frames because this paper is devoted to completeness results; using only standard frames typically leads to logics that are not axiomatisable. For example, if there are no propositional quantifiers in the language, the basic modal logic **K** (the set of basic modal formulas valid on all frames) is decidable in PSPACE; but if we add propositional quantifiers — and interpret them using only *standard* frames — the resulting set of validities is not even recursively enumerable.[4] Modal languages with propositional quantifiers may look simple, but (interpreted over standard frames) they are powerful second-order systems.

Leon Henkin [12] introduced a way of "taming" higher-order logics. The underlying idea is to increase the number of models, thereby reducing the number

[4] This negative result (and others) were proved in Kit Fine's pioneering 1970 paper [11] (though the paper also contains an interesting positive result: extending ordinary **S5** with propositional quantification using standard frames yields a decidable logic). For some sharper negative results see [13].

of validities — hopefully to the point where the set of validities becomes recursively enumerable. General frames can be viewed as a (successful) Henkin-style attempt to tame modal semantics: a general frame is just a Kripke frame $\langle W, R\rangle$ together with a selection of propositions Π, that is, subsets of W.[5] We don't insist that all subsets of W belong to Π; we simply insist that Π has enough structure to behave like a set of propositions. In particular, in any general frame, the set of admissible propositions should be closed under the operations corresponding to the boolean operators and $\Box$. So there are a lot more general frames than frames — a single Kripke frame gives rise to multiple general frames — and it turns out that this expansion successfully "tames" the set of validities.

The success of general frames in the setting of ordinary modal logic leads to the questions that drive this paper. What happens if we add propositional quantification to basic *hybrid* logic rather than just basic modal logic? In particular: what is the logic of general frames if our base language contains not only booleans and boxes, but also nominals and satisfaction operators? Moreover, do general frames allow us to "tame" not merely the basic logic, but also what hybrid logicians call its pure extensions?

But this is jumping ahead. We must first answer a more basic question: how do we interpret $\mathcal{L}_{BHPQ}$ on general frames? The interpretation of the nominals will be taken care of by a naming function (or *nomination*) N assigning a world to each nominal, while the interpretation of the propositional letters will be given by a modal valuation function V.

Definition 2 (General models and standard models). *A* general model $\mathcal{M}$ based on *a general frame* $\langle W, R, \Pi\rangle$ *is a tuple* $\langle W, R, \Pi, N, V\rangle$ *where* $N : NOM \to W$ *and* $V : PLET \to \Pi$. *A* standard model *is a model based on a standard frame.*

This definition builds in our central semantic design decision for $\mathcal{L}_{BHPQ}$: *the interpretation of the nominals is independent of the choice of* Π. Nominals directly "tag" arbitrary worlds via the nomination. This is important, because Π may not contain all — or even any — singleton subsets of W as admissible propositions. Our nominals ignore Π. They are hardwired to the underlying Kripke frame.

Definition 3 (Variable assignments and variants). *A* variable assignment *on a general frame* $\mathcal{G} = \langle W, R, \Pi\rangle$ *is a function* $g : PVAR \to \Pi$. *For any propositional variable* p, *we say that a variable assignment* g' *is a* p-variant *of variable assignment* g *iff for all propositional variables* $q \neq p$, *we have* $g'(q) = g(q)$. *We write this as* $g' \sim_p g$.

[5] There is more to general frames that this: they can also be viewed as representations of modal algebras; see Chapter 5 of [6] for details. Both lines of work stem from a classic paper by S. K. Thomason [21]. This links general frames and modal algebras, and shows that (a) there are Priorean tense logics that are not complete with respect to any class of frames, that (b) every Priorean tense logic is complete with respect to a class of general frames. That is: general frames "tame" frame validity.

Now for a Henkin-style definition of satisfaction and truth:

Definition 4 (Satisfaction and truth). *Let $\mathcal{M} = \langle W, R, \Pi, N, V\rangle$ be a general model, and g be a variable assignment on $\langle W, R, \Pi\rangle$. We define what it means for $\mathcal{M}$ to satisfy a formula at a world w with respect to assignment g as follows:*

- $\mathcal{M}, g, w \models i$ *iff* $w = N(i)$, *for any* $i \in NOM$
- $\mathcal{M}, g, w \models c$ *iff* $w \in V(c)$, *for any* $c \in PLET$
- $\mathcal{M}, g, w \models p$ *iff* $w \in g(p)$, *for any* $p \in PVAR$
- $\mathcal{M}, g, w \models \neg\varphi$ *iff* $\mathcal{M}, g, w \not\models \varphi$
- $\mathcal{M}, g, w \models \varphi \wedge \psi$ *iff* $\mathcal{M}, g, w \models \varphi$ *and* $\mathcal{M}, g, w \models \psi$
- $\mathcal{M}, g, w \models \Box\varphi$ *iff for all* $v \in W$, *if* wRv *then* $\mathcal{M}, g, v \models \varphi$
- $\mathcal{M}, g, w \models @_i\varphi$ *iff* $\mathcal{M}, g, N(i) \models \varphi$
- $\mathcal{M}, g, w \models \forall p\varphi$ *iff for all* $g' \sim_p g$, *we have* $\mathcal{M}, g', w \models \varphi$.

A formula φ is true *at a world w in $\mathcal{M}$ iff for all variable assignments g, $\mathcal{M}, g, w \models \varphi$, and we write this as $\mathcal{M}, w \models \varphi$.*

Definition 5 (Validity and consequence). *A formula φ is* valid in *a general model $\mathcal{M}$ iff it is true at all worlds in $\mathcal{M}$; we write this as $\mathcal{M} \models \varphi$. A formula φ is* valid *iff it is valid in all general models; and we write this as $\models \varphi$.*

A formula φ is a consequence of a set of formulas Γ, written $\Gamma \models \varphi$, iff for all general models $\mathcal{M}$, all assignments g on $\mathcal{M}$, and all worlds w in $\mathcal{M}$, if $\mathcal{M}, g, w \models \Gamma$ then $\mathcal{M}, g, w \models \varphi$. Here $\mathcal{M}, g, w \models \Gamma$ means: for all formulas $\gamma \in \Gamma$, $\mathcal{M}, g, w \models \gamma$. Note: φ is valid iff $\emptyset \models \varphi$.

We could also have defined notions of *standard validity* and *standard consequence*; these are defined exactly as above but with "standard model(s)" replacing "general model(s)". But, as discussed earlier, for the purposes of the present paper standard structures are of little interest. Completeness results are rare when working with standard structures, but by working with general models we will be able to prove Henkin-style completeness results that cover both the basic logic and all its pure extensions (we will explain this terminology later).

The following semantic lemmas will be used in our soundness and completeness proofs. We start with the Agreement Lemma. This tells us that to ensure that nominations, valuations, and assignments agree on whether φ is satisfied, it suffices that they agree on the atomic symbols actually occurring in φ

Lemma 1 (Agreement Lemma). *Let φ be a formula, and let both $\mathcal{M} = \langle W, R, \Pi, N, V\rangle$ and $\mathcal{M}^* = \langle W, R, \Pi, N^*, V^*\rangle$ be general models based on $\langle W, R, \Pi\rangle$ such that:*

i) $V(c) = V^*(c)$ for all propositional letters c occurring in φ, and

ii) $N(i) = N^*(i)$ for all nominals i occurring in φ.

Furthermore, let g and h be variable assignments on $\langle W, R, \Pi\rangle$ such that $g(q) = h(q)$ for all the free propositional variables q occurring in φ. Then for all $w \in W$, we have that $\mathcal{M}, g, w \models \varphi$ iff $\mathcal{M}^*, h, w \models \varphi$.

Proof. By induction of the number of propositional connectives in φ. $\Box$

A standard corollary follows: the variable assignment is irrelevant when evaluating sentences, so for sentences φ can write $\mathcal{M}, w \models \varphi$ instead of $\mathcal{M}, g, w \models \varphi$.

Definition 6. *Let $\mathcal{M} = \langle W, R, \Pi, N, V\rangle$ be a general model, and g an assignment on $\langle W, R, \Pi\rangle$. Then for all formulas φ we define*

$$[\mathcal{M}, g]_\varphi = \{w \in W \mid \mathcal{M}, g, w \models \varphi\}.$$

For φ a sentence we can just write $[\mathcal{M}]_\varphi$, as g is irrelevant.

Next we see that all soft-QF formulas pick out admissible propositions.

Lemma 2. *Let $\mathcal{M} = \langle W, R, \Pi, N, V\rangle$ be a general model, and g be any assignment on $\langle W, R, \Pi\rangle$. Then for all soft-QF formulas φ, we have $[\mathcal{M}, g]_\varphi \in \Pi$.*

Proof. By induction on the number of connectives in soft-QF formulas. All propositional letters are assigned elements of Π by V, and any assignment g on $\mathcal{M}$ assigns all propositional variables an element of Π, which establishes the base case. The inductive steps for $\neg\varphi$, $\varphi \wedge \psi$, and $\Box\varphi$, follow from the three closure conditons on Π. As for the $@_i\varphi$ step, note that any such formula is either true at all worlds, or false at all worlds, that is any such formula picks out either the proposition W or $\emptyset$. But these two propositions are always admissible: as Π is non-empty, it contains at least one proposition X. But then $\emptyset = X \cap (W - X)$ and $W = W - \emptyset$ are both in Π $\Box$

Our next lemma tells us that the set of all propositions picked out by soft-QF formulas is a subset of Π that is a closed collection of admissible propositions. Let us write $[\mathcal{M}, g]^{sqf}$ for $\{[\mathcal{M}, g]_\varphi : \varphi$ *is a soft-QF formula*$\}$. Then:

Lemma 3. *Given any general model $\mathcal{M} = \langle W, R, \Pi, N, V\rangle$ and an assignment g on $\mathcal{M}$:*

1. *$[\mathcal{M}, g]^{sqf} \subseteq \Pi$, and*
2. *$[\mathcal{M}, g]^{sqf}$ is a closed set of admissible propositions.*

Proof. Item 1 follows from Lemma 2. Item 2 holds because the three closure conditions conditions correspond to the connectives $\neg$, $\wedge$ and $\Box$. Argue as follows:
Relative complement: Consider $[\mathcal{M}, g]_\varphi$ for some soft-QF formula φ. It is sufficient to show that $\Pi - [\mathcal{M}, g]_\varphi = [\mathcal{M}, g]_{\neg\varphi}$. But $w \in [\mathcal{M}, g]_{\neg\varphi}$ iff $\mathcal{M}, g, w \models \neg\varphi$ iff $\mathcal{M}, g, w \not\models \varphi$ iff $w \in \Pi - [\mathcal{M}, g]_\varphi$.
Intersection: Similar to the previous case.
Modal projection: We show that $\{w \in W \mid \forall v(wRv \to v \in [\mathcal{M}, g]_\varphi)\} = [\mathcal{M}, g]_{\Box\varphi}$ for any soft-QF formula φ. But $u \in [\mathcal{M}, g]_{\Box\varphi}$ iff $\mathcal{M}, g, u \models \Box\varphi$ iff $\forall v(uRv \to v \in [\mathcal{M}, g]_\varphi)$ iff $u \in \{w \in W \mid \forall v(wRv \to v \in [\mathcal{M}, g]_\varphi)\}$. $\Box$

Now for the Substitution Lemma; note the restriction to soft-QF formulas.

Lemma 4 (Substitution Lemma). *Let $\mathcal{M} = \langle W, R, \Pi, N, V\rangle$ be a general model and let g be a variable assignment on $\langle W, R, \Pi\rangle$. Then for any safe substitution $\varphi[\psi/p]$, where ψ is a soft-QF formula, we have that:*

$$\mathcal{M}, g, w \models \varphi[\psi/p] \text{ iff } \mathcal{M}, g', w \models \varphi,$$

where $g' \sim_p g$ is defined by setting $g'(p) = [\mathcal{M}, g]_\psi$.

Proof. First note that g' is well-defined since $[\mathcal{M}, g]_\psi \in \Pi$ by the previous lemma. The proof is by induction on the number of connectives in φ. The interesting case is $\varphi = \forall q\theta$. We have three subcases:
If $p = q$, then the result follows from the Agreement Lemma.
If $p \neq q$, but p does not occur free in θ, then the result again follows from the Agreement Lemma.
Finally there is the case where $p \neq q$ and p occurs free in θ. Then $\mathcal{M}, g, w \models (\forall q\theta)[\psi/p]$ iff $\mathcal{M}, g, w \models \forall q(\theta[\psi/p])$ iff for all $g'' \sim_q g$, we have $\mathcal{M}, g'', w \models \theta[\psi/p]$. But by the induction hypothesis this is equivalent to for all $g'' \sim_q g$, we have $\mathcal{M}, g''', w \models \theta$ where $g''' \sim_p g''$ is defined by setting $g'''(p) = [\mathcal{M}, g'']_\psi$. But giving a g''' such that $g''' \sim_p g''$ where $g'''(p) = [\mathcal{M}, g'']_\psi$ and $g'' \sim_q g$ is equivalent to giving a g''' such that $g''' \sim_q g'$ where $g' \sim_p g$ is defined by setting $g'(p) = [\mathcal{M}, g'']_\psi$. But q cannot occur free in ψ, so $[\mathcal{M}, g'']_\psi = [\mathcal{M}, g]_\psi$ by the Agreement Lemma. The result follows from $\mathcal{M}, g', w \models \forall q\theta$ being equivalent to for all $g''' \sim_q g'$, so we have $\mathcal{M}, g''', w \models \theta$. □

4 The axiomatisation

Our axiomatisation is called $\mathbf{K}_{SQph}$+RULES. It is an extension of the $\mathbf{K}_h$+RULES axiomatisation for basic propositional hybrid logic presented in Chapter 7 Section 3 of [6]. We first present the two components of $\mathbf{K}_h$+RULES axiomatisation, and then add on what we need to handle propositional quantification.

Definition 7 (The $\mathbf{K}_h$ axiom system). *The $\boldsymbol{K}_h$ axiom system contains as axioms all propositional tautologies, and all instances of K for the modalities:*

$K_\Box$: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$K_@$: $@_i(\varphi \to \psi) \to (@_i\varphi \to @_i\psi)$.

It also contains all instances of the following interaction schemas:

Intro: $i \wedge \varphi \to @_i\varphi$
Agree: $@_j @_i\varphi \leftrightarrow @_i\varphi$
Back: $\Diamond @_i\varphi \to @_i\varphi$
Sdual: $@_i\varphi \leftrightarrow \neg @_i \neg\varphi$,

and in addition, all instances of the modal equality schemas:

Ref: $@_i i$
Sym: $@_i j \leftrightarrow @_j i$
Nom: $@_i j \wedge @_j p \to @_i\varphi$.

The proof rules of $\mathbf{K}_h$ *are:*

MP: If $\vdash \varphi \to \psi$ *and* $\vdash \varphi$ *then* $\vdash \psi$
Gen$_\Box$*: If* $\vdash \varphi$ *then* $\vdash \Box\varphi$
Gen$_@$*: If* $\vdash \varphi$ *then* $\vdash @_i\varphi$

$\mathbf{K}_h$ proofs are Hilbert-style proofs and it is fairly straighforward to adapt the usual modal machinery of canonical models and prove that $\mathbf{K}_h$ is a (sound and) strongly complete axiom system for minimal propositional hybrid logic (see Chapter 7, Section 3 of [6] for details). The interaction axioms (together with the Gen$_@$ rule) capture the logic of the satisfaction operators: self-dual normal modal operators that interact smoothly with the other connectives. The axioms *Ref*, *Sym*, and *Nom* capture the logic of atomic satisfaction statements like $@_i j$; such statements are "modal equality assertions", modal equivalents of first-order atomic formulas of the form $i = j$. Clearly *Ref* and *Sym* express the reflexivity and symmetry of identity. The *Nom* axiom is more interesting. It can be read as a Leibniz-style identity axiom: "if i and j are identical, and i has property φ, then j has property φ too". But also note an important special case: if φ is k this becomes $@_i j \wedge @_j k \to @_i k$, which expresses the transitivity of identity.

Here are two schemas that are used in the completeness proof:

Elim: $i \wedge @_i\varphi \to \varphi$
Bridge: $\Diamond i \wedge @_i\varphi \to \Diamond\varphi$

Note that *Elim* is a contraposed form of the *Intro* axiom (using the *Sdual* axiom). As for *Bridge*, here are the main steps of a $\mathbf{K}_h$ proof of it:

1) $\Diamond i \wedge \Box\varphi \to \Diamond(i \wedge \varphi)$	*Modal validity*
2) $i \wedge \varphi \to @_i\varphi$	*Intro axiom*
3) $\Box(i \wedge \varphi \to @_i\varphi)$	Gen$_@$ *on 2*
4) $\Box(i \wedge \varphi \to @_i\varphi) \to (\Diamond(i \wedge \varphi) \to \Diamond @_i\varphi)$	*Modal validity*
5) $\Diamond(i \wedge \varphi) \to \Diamond @_i\varphi$	*3,4 Modus ponens*
6) $\Diamond i \wedge \Box\varphi \to \Diamond @_i\varphi$	*1,5 Propositional logic*
7) $\Diamond @_i\varphi \to @_i\varphi$	*Back axiom*
8) $\Diamond i \wedge \Box\varphi \to @_i\varphi$	*6,7 Propositional logic*
9) $\Diamond i \wedge @_i\varphi \to \Diamond\varphi$	*8 Contraposition, Sdual axiom*

Nonetheless, despite the fact that $\mathbf{K}_h$ is complete with respect to the class of all Kripke models (that is, it is the "minimal hybrid logic"), it is more usual to work with more powerful proof systems such as $\mathbf{K}_h$+RULES.[6]

Definition 8 (The $\mathbf{K}_h$+RULES axiom system). *The* $\mathbf{K}_h$+RULES *axiom system contains all the axioms and rules of* $\mathbf{K}_h$ *plus the following two proof rules:*

$$Name: \frac{\vdash j \to \theta}{\vdash \theta} \qquad Paste: \frac{\vdash @_i\Diamond j \wedge @_j\varphi \to \theta}{\vdash @_i\Diamond\varphi \to \theta}$$

In both rules, j is a nominal distinct from i that does not occur in φ or θ.

[6] Several such systems have been explored; see [1,5] for more. Here we follow [6].

As we shall see later, these two rules allow us to do some things that are *not* possible in $\mathbf{K}_h$ — things that will become important when we look at the pure extensions of $\mathbf{K}_h$. Anticipating this, we shall define $\mathbf{K}_{SQph}$+RULES, our basic axiomatisation for minimal propositional hybrid logic with propositional quantification, on top of $\mathbf{K}_h$+RULES.

Definition 9 (The $\mathbf{K}_{SQph}$+RULES axiom system). *The* $\boldsymbol{K}_{SQph}$+RULES *axiom system contains all the axioms and rules of* $\boldsymbol{K}_h$+RULES. *It also contains the following axioms:*

Q1: $\forall p(\varphi \to \psi) \to (\varphi \to \forall p\psi)$, *where* φ *contains no free occurrences of* p
Q2-sqf: $\forall p\varphi \to \varphi[\psi/p]$, *where* $\varphi[\psi/p]$ *is a soft-QF substitution*
Barcan$_{@}$*:* $\forall p @_i\varphi \leftrightarrow @_i\forall p\varphi$,

and one more proof rule:

Gen$_\forall$*: If* $\vdash \varphi$ *then* $\vdash \forall p\varphi$.

While *Q1* is familiar from modal and classical logic, and *Barcan*$_{@}$ from hybrid logic, the side condition on *Q2-sqf* deserves comment We cannot substitute nominals (as not all admissible sets of propositions contain singleton subsets) nor can we substitute quantified formulas either (our Substitution Lemma does not cover such substitutions, since the admissible sets are only required to be closed under finite intersections, not arbitrary intersections). Allowing only soft-QF substitutions ensures soundness.[7]

Definition 10 (Provability and consistency in $\mathbf{K}_{SQph}$+RULES). *A formula* φ *is* $\mathbf{K}_{SQph}$ + RULES-provable *iff there is a* $\mathbf{K}_{SQph}$ + RULES *Hilbert-style proof of* φ*; we write* $\vdash \varphi$ *for provability and* $\nvdash \varphi$ *for unprovability. A formula* φ *is* $\mathbf{K}_{SQph}$ + RULES-provable from a set of formulas Σ *iff for some conjunction* σ *of formulas from* Σ *we have* $\vdash \sigma \to \varphi$. *A formula* φ *is* $\mathbf{K}_{SQph}$ + RULES-consistent *iff* $\nvdash \neg\varphi$. *A set of formulas* Σ *is* $\mathbf{K}_{SQph}$ + RULES-consistent *iff there is no conjunction* σ *of formulas from* Σ *such that* $\vdash \neg\sigma$.

Theorem 1. $\boldsymbol{K}_{SQph}$+RULES *is sound with respect to general frames.*

Proof. To prove this we need to show that (a) all the axioms are valid on all general frames, and (b) that the proof rules preserve validity. This is known for all the axioms and rules in K_h+RULES, so we only need to check that the *Q1* and *Q2-sqf* are valid and that *Gen*$_\forall$ preserves validity. That *Gen*$_\forall$ preserves validity is more-or-less immediate. The axioms are also easy to handle; we present the argument for *Q2-sqf* and leave *Q1* and *Barcan*$_{@}$ for the reader.

[7] Note that on *standard* models we could drop the restriction prohibiting substitution of quantified formulas as standard models admit all subsets of the frame as propositions. That is, the rule which permits any soft substitution is sound on standard models. However this does not lead to a completeness result for standard models, as (thanks to Kit Fine's results [11]) we know that the set of all standard validities on the class of all standard models is not recursively enumerable.

So: choose an arbitrary general model $\mathcal{M} = \langle W, R, \Pi, N, V \rangle$, let $w \in W$, and let g be a variable assignment on $\langle W, R, \Pi \rangle$. Then to show that $\forall p\varphi \rightarrow \varphi[\psi/p]$ is valid, where $\varphi[\psi/p]$ is a soft-QF substitution, suppose that $\mathcal{M}, g, w \models \forall p\varphi$. This is equivalent to: for all g' such that $g' \sim_p g$, we have $\mathcal{M}, g', w \models \varphi$. Define $g'' \sim_p g$ by setting $g''(p) = [\mathcal{M}, g]_\psi$; Lemma 2 tells us that g'' is a well-defined variant of g because ψ is a soft-QF formula. Hence $\mathcal{M}, g'', w \models \varphi$ and so, using the Substitution Lemma, $\mathcal{M}, g, w \models \varphi[\psi/p]$.

Lemma 5. *If* $\vdash \sigma \rightarrow \theta[c/p]$ *and* c *does not occur in* θ *or* σ*, then* $\vdash \sigma \rightarrow \forall q\theta[q/p]$*, where* q *is any variable not occurring in* θ *or* σ*.*

Proof. Left to the reader. □

Lemma 6. *Suppose that* q *can be safely substituted for* p *in* φ *and that* φ *has no free occurrences of* q*. Then* $\vdash \forall p\varphi \leftrightarrow \forall q\varphi[q/p]$*.*

Proof. Left to the reader. □

One final remark. We formulated *Barcan*$_@$ in its $\forall$-form, that is, in the form $\forall p@_i\varphi \leftrightarrow @_i\forall p\varphi$. Its $\exists$-form is $@_i\exists p\varphi \leftrightarrow \exists p@_i\varphi$. Strictly speaking, the left-to-right arrows of both forms are Barcan formulas, while the right-to-left directions of both are converse Barcan formulas.

5 Strong Completeness

We will now extend a standard model-building strategy used to prove the strong completeness of $\mathbf{K}_h$ + RULES (see Chapter 7, Section 3 of [6]) to show that every $\mathbf{K}_{SQph}$ + RULES-consistent set of formulas has a general model. The general model we shall construct has a number of special properties (described below) which will enable us to prove strong completeness not only for $\mathbf{K}_{SQph}$ + RULES itself, but for all of its pure extensions as well.

Definition 11 ($\mathbf{K}_{SQph}$ + RULES Maximal Consistent Sets). *Fix a language of* $\mathcal{L}_{BHPQ}$*. A set of formulas* Σ *in this language is a* $\mathbf{K}_{SQph}$ + RULES-MCS *iff it is* $\mathbf{K}_{SQph}$ + RULES*-consistent, and any proper extension (in the same language) is inconsistent.*

Lemma 7 (Named sets yielded by an MCS). *Let* Γ *be a* $\mathbf{K}_{SQph}$ + RULES-*MCS. For every nominal* i*, let* $\Delta_i = \{\varphi \mid @_i\varphi \in \Gamma\}$*. Then:*

1. *For every nominal* i*,* Δ_i *is a* $\mathbf{K}_{SQph}$ + RULES-MCS *that contains* i*.*
2. *For all nominals* i *and* j*, if* $i \in \Delta_j$*, then* $\Delta_j = \Delta_i$*.*
3. *For all nominals* i *and* j*,* $@_i\varphi \in \Delta_j$ *iff* $@_i\varphi \in \Gamma$*.*
4. *If a nominal* $k \in \Gamma$*, then* $\Gamma = \Delta_k$*.*

Proof. This is Lemma 7.24 of [6], and proof details can be found there. In fact, Lemma 7.24 shows that only $\mathbf{K}_h$ reasoning is needed to prove this lemma. □

The Δ_i in this lemma are called the *named sets yielded by* Γ. Clause 1 of the previous lemma tells us that each of these is an MCS containing at least one nominal; all of these MCSs are "hidden inside" the original $\mathbf{K}_h$-MCS. Clause 2 tells us that each nominal picks out a unique such MCS, and Clause 3 tells us that any satisfaction statement is either in all the Δ_j or in none of them. So every $\mathbf{K}_h$-MCS contains *almost* all that is required to build structures in which *every world is named by some nominal*. But not quite. For note that Clause 4 is only a conditional — we have no guarantee that Γ itself is one of the MCSs "hidden inside" Γ.

Indeed, this weakness in Clause 4 is the key reason for using $\mathbf{K}_h$+RULES instead of $\mathbf{K}_h$. The addition of the *Name* and *Paste* rules *does* allow us to guarantee that Γ itself is one of the MCSs "hidden inside" Γ. This will let us create a "named and pasted" MCS, which contains all the information required to build a frame in which each world is named by a nominal, which will prove crucial for the pure extensions completeness result. Furthermore, the axioms and rules for propositional quantification in $\mathbf{K}_{SQph}$ + RULES also ensure that this MCS is "witnessed", which allows us to define a suitable set of admissible propositions over its frame, thereby creating a general model. We first define what we mean by "named", "pasted" and "witnessed" and then prove the Lindenbaum-style lemma which will lead us to these goals.

Definition 12 (Named, pasted and witnessed MCSs). *Let Σ be a $\mathbf{K}_{SQph}$+ RULES-MCS. Then we say:*

- Σ *is* named *iff for some nominal i, $i \in \Sigma$,*
- Σ *is* pasted *iff for every formula of the form $@_i\Diamond\varphi \in \Sigma$, there is some nominal j such that $@_i\Diamond j \in \Sigma$ and $@_j\varphi \in \Sigma$, and*
- Σ *is* witnessed *iff for every formula of the form $@_i\exists p\varphi$, there is some propositional letter c such that $@_i\varphi[c/p] \in \Sigma$.*

To prove a Lindenbaum-style lemma for $\mathbf{K}_{SQph}$+RULES, we must extend the language. Suppose we start with language $\mathcal{L}$. We will add a countably infinite set of nominals (called *NewN*) and a countably infinite set of new propositional letters (called *NewL*), and call the extended language $\mathcal{L}'$. We will use *NewN* for naming and pasting and *NewL* for witnessing.

Lemma 8 (Lindenbaum). *Every $\mathbf{K}_{SQph}$ + RULES-consistent set of formulas in language $\mathcal{L}$ can be extended to a named, pasted and witnessed $\mathbf{K}_{SQph}$ + RULES-MCS in language $\mathcal{L}'$.*

Proof. Given a consistent set of $\mathcal{L}$-formulas Σ, add *NewN* and *NewP* as just described to form $\mathcal{L}'$, and enumerate all three sets. Define Σ_k to be $\Sigma \cup \{k\}$, where k is the first nominal in *NewN*. Σ_k is consistent. For suppose not. Then for some conjunction of formulas θ from Σ, we have $\vdash k \to \neg\theta$. But k is a new nominal, so it does not occur in θ; hence, by the NAME rule we have $\vdash \neg\theta$. This contradicts the consistency of Σ, so Σ_k must be consistent.

Define Σ^0 to be Σ_k, and suppose we have defined Σ^m, where $m \geq 0$. Let φ_{m+1} be the $(m+1)$-th formula in our enumeration of $\mathcal{L}'$. We define Σ^{m+1} as follows:

If $\Sigma^{m+1} \cup \{\varphi_{m+1}\}$ is inconsistent, then $\Sigma^{m+1} = \Sigma^m$. Otherwise:

1. $\Sigma^{m+1} = \Sigma^m \cup \{\varphi_{m+1}\}$, if φ_{m+1} is not of the form $@_i\Diamond\varphi$ or $@_i\exists p\varphi$.
2. $\Sigma^{m+1} = \Sigma^m \cup \{\varphi_{m+1}\} \cup \{@_i\Diamond j \wedge @_j\varphi\}$, if φ_{m+1} is of the form $@_i\Diamond\varphi$. (Here j is the first nominal in the enumeration of *NewN* that does not occur in Σ^m or $@_i\Diamond\varphi$).
3. $\Sigma^{m+1} = \Sigma^m \cup \{\varphi_{m+1}\} \cup \{@_i\varphi[c/p]\}$, if φ_{m+1} is of the form $@_i\exists p\varphi$. (Here c is the first propositional letter in the enumeration of *NewL* that does not occur in Σ^m or $@_i\exists p\varphi$).

Let $\Sigma^* = \bigcup_{n\geq 0}\Sigma^n$. Clearly this set is named (as we added k in the first step), maximal (by construction), pasted and witnessed.

Furthermore, it is consistent, for the only aspects of the expansion that require checking are those given in by the second and third steps. Preservation of consistency by the second step is precisely what the PASTE rule guarantees. As for the third step, we argue as follows. Assume for the sake of contradiction that $\Sigma^m \cup \{@_i\exists p\varphi\} \cup \{@_i\varphi[c/p]\}$ is inconsistent. Then there are formulas $\sigma_1, ..., \sigma_n$ in Σ^m such that $\vdash \neg(\sigma_1 \wedge ... \wedge \sigma_n \wedge @_i\exists p\varphi \wedge @_i\varphi[c/p])$. Writing $\sigma_1 \wedge ... \wedge \sigma_n$ as σ, propositional logic yields $\vdash (\sigma \wedge @_i\exists p\varphi) \rightarrow \neg @_i\varphi[c/p]$. The conditions of Lemma 5 are fulfilled, so we have $\vdash (\sigma \wedge @_i\exists p\varphi) \rightarrow \forall q\neg @_i\varphi[q/p]$ where q is a new propositional variable. This is equivalent to $\vdash (\sigma \wedge @_i\exists p\varphi) \rightarrow \neg\exists q @_i\varphi[q/p]$, so using (the $\exists$-form of) $Barcan_{@}$ we have $\vdash (\sigma \wedge @_i\exists p\varphi) \rightarrow \neg @_i\exists q\varphi[q/p]$. Moreover, the conditions of Lemma 6 are fulfilled as well, so $\vdash @_i\exists p\varphi \leftrightarrow @_i\exists q\varphi[q/p]$, and we have that $\vdash (\sigma \wedge @_i\exists p\varphi) \rightarrow \neg @_i\exists p\varphi$. This contradicts the consistency of $\Sigma^m \cup \{@_i\exists p\varphi\}$, thus the third step must preserve consistency after all. □

Lemma 9 (Generated admissible sets). *Let $\mathcal{F} = \langle W, R\rangle$ be a Kripke frame, let $\mathcal{P}(W)$ be the powerset of W, let $B \subseteq \mathcal{P}(W)$ and define $\Pi(B)$, the* admissible set generated by B*, to be the smallest subset of $\mathcal{P}(W)$ containing B that is closed under relative complement, intersection, and modal projection. Then $\langle W, R, \Pi(B)\rangle$ is a general frame.*

Proof. Immediate by definition of $\Pi(B)$. □

Lemma 10. *Let $\mathcal{F} = \langle W, R\rangle$ be a Kripke frame, and let N be any nomination on $\mathcal{F}$. Let V be any mapping such that $V : PLET \rightarrow \mathcal{P}(W)$ and let g be any mapping such that $g : PVAR \rightarrow \mathcal{P}(W)$. Then $\mathcal{M} = \langle W, R, \Pi(im(V) \cup im(g)), N, V\rangle$ is a general model (here im(V) and im(g) are the images of V and g respectively).*

Proof. As $\langle W, R\rangle$ is a Kripke frame and $im(V) \cup im(g) \subseteq \mathcal{P}(W)$, by the previous lemma $\langle W, R, \Pi(im(V) \cup im(g))\rangle$ is a general frame. Hence V and g are mappings into $\Pi(im(V) \cup im(g))$, thus V is a valuation and g is an assignment on a general frame. Thus $\mathcal{M}$ is a general model. □

Definition 13 (Canonical named structures). *Let Γ be a named, pasted and witnessed $\mathbf{K}_{SQph}$ + RULES-MCS. Let $\mathcal{M}^\Gamma$ be $\langle W^\Gamma, R^\Gamma, \Pi^\Gamma, N^\Gamma, V^\Gamma\rangle$ where:*

- *W^Γ is the set of all named sets yielded by Γ.*
- *R^Γ is the standard modal canonical relation between MCSs. That is, for any $u, v \in W^\Gamma$ we define $uR^\Gamma v$ iff for all formulas φ, $\varphi \in v$ implies $\Diamond\varphi \in u$. Or equivalently: $uR^\Gamma v$ iff for all formulas φ, $\Box\varphi \in u$ implies $\varphi \in v$.*
- *$N^\Gamma : NOM \to W^\Gamma$ is defined as follows. For any any nominal i, $N(i)$ is the unique $w \in W^\Gamma$ such that $i \in w$; that is, $N(i) = \Delta_i$.*
- *$V^\Gamma : PLET \to \mathcal{P}(W^\Gamma)$ is the standard modal canonical valuation (for proposition letters). That is, $V^\Gamma(c) = \{w \in W^\Gamma \mid c \in w\}$, for any proposition letter c.*
- *$g^\Gamma : PVAR \to \mathcal{P}(W^\Gamma)$ is the standard modal canonical valuation (for proposition variables). That is, $g^\Gamma(p) = \{w \in W^\Gamma \mid p \in w\}$, for any proposition variable p.*
- *Π^Γ is $\Pi(im(V^\Gamma) \cup im(g^\Gamma))$.*

We now check that this definition does indeed gives rise to Kripke frames and general models where every world is named by some nominal.

Lemma 11. *Let Γ be a named, pasted and witnessed $\mathbf{K}_{SQph}$ + RULES-MCS, and let $\mathcal{M}^\Gamma$ be the* canonical named general model *yielded by Γ. Then $\langle W^\Gamma, R^\Gamma\rangle$ is a Kripke frame and $\mathcal{M}^\Gamma$ is a named general model.*

Proof. To see that $\langle W^\Gamma, R^\Gamma\rangle$ is a Kripke frame, first note that by Lemma 7 (1), for every nominal i, Δ_i is a $\mathbf{K}_{SQph}$ + RULES-MCS containing i. As W^Γ is a non-empty set of MCSs, the standard modal canonical relation R^Γ can be defined over it, thus $\langle W^\Gamma, R^\Gamma\rangle$ is a Kripke frame. Moreover N^Γ is a well-defined nomination, for Lemma 7 (2) guarantees that Δ_i is the unique element of W^Γ such that $i \in w$, and it clearly "names" every world in W^Γ. Finally, both V^Γ and g^Γ are well-defined, so we have all we need to apply Lemma 10 and conclude that $\mathcal{M}^\Gamma$ is a named general model. □

So from now on we will call $\langle W^\Gamma, R^\Gamma\rangle$ the *canonical named Kripke frame* yielded by Γ, and $\mathcal{M}^\Gamma$ the *canonical named general model* yielded by Γ. We now examine them more closely. Our first lemma tells us that R^Γ works exactly as it does in ordinary propositional modal logic.

Lemma 12 (Existence Lemma). *Let Γ be a named, pasted, and witnessed $\mathbf{K}_{SQph}$ + RULES-MCS, and let $\mathcal{M}^\Gamma$ be the canonical named general model yielded by Γ. Suppose $u \in W^\Gamma$ and $\Diamond\varphi \in u$. Then there is a $v \in W^\Gamma$ such that $uR^\Gamma v$ and $\varphi \in v$.*

Proof. This is essentially Lemma 7.27 from [6]. □

Lemma 13. *Let Γ be a named, pasted and witnessed $\mathbf{K}_{SQph}$ + RULES-MCS, let $\mathcal{M}^\Gamma$ be the canonical named general model yielded by Γ, and let $u \in W^\Gamma$. Then for all quantifier-free formulas φ, we have that:*

1. $\mathcal{M}^\Gamma, g^\Gamma, u \models \varphi$ *iff* $\varphi \in u$
2. $\mathcal{M}^\Gamma, g^\Gamma, \Delta_i \models \varphi$ *iff* $@_i\varphi \in \Gamma$.

Proof. Item 1 follows by induction on the number of connectives. It is essentially Lemma 7.28 from [6]. Item 2 then follows by the definition of Δ_i. □

Next for a simple but important lemma:

Lemma 14. *Let Γ be a named, pasted and witnessed* $\mathbf{K}_{SQph}$ + RULES-*MCS, let $\mathcal{M}^\Gamma$ be the canonical named general model yielded by Γ. Then* $[\mathcal{M}^\Gamma, g^\Gamma]^{sqf} = \Pi^\Gamma$.

Proof. Item 1 of Lemma 3 tells us that $[\mathcal{M}^\Gamma, g^\Gamma]^{sqf} \subseteq \Pi^\Gamma$. Item 2 of same lemma tells us that $[\mathcal{M}^\Gamma, g^\Gamma]^{sqf}$ is a closed set of admissible propositions. Now, $im(V^\Gamma) \cup im(g^\Gamma) \subseteq [\mathcal{M}, g]^{sqf}$, as these are the atomic propositions picked out by the propositional constants and variables. But Π^Γ is $\Pi(im(V^\Gamma) \cup im(g^\Gamma))$, the *smallest* closed set of admissible propositions containing $im(V^\Gamma) \cup im(g^\Gamma)$. So $\Pi^\Gamma \subseteq [\mathcal{M}^\Gamma, g^\Gamma]^{sqf}$. □

The previous lemma is important because it gives us a *syntactic* handle on the elements of Π^Γ: every proposition in Π^Γ is “picked out” by some soft-QF formula; this syntactic characterisation enables us to prove the final lemma leading to completeness.

Lemma 15 (Truth Lemma). *Let Γ be a named, pasted, and witnessed* $\mathbf{K}_{SQph}$+ RULES-*MCS, and let $\mathcal{M}^\Gamma$ be the canonical named general model yielded by Γ, and let $u \in W^\Gamma$. Then, for all formulas φ, we have that $\mathcal{M}^\Gamma, g^\Gamma, u \models \varphi$ iff $\varphi \in u$.*

Proof. For any formula φ, let $con(\varphi)$ be the number of connectives in φ. Moreover, let $quan(\varphi)$ be the maximal depth of quantifier nesting in φ, that is:

$$quan(\varphi) = \begin{cases} 0 & \text{if } \varphi \text{ is atomic} \\ sup\{quan(\psi), quan(\theta)\} & \text{if } \varphi = \psi \wedge \theta \\ quan(\psi) & \text{if } \varphi \in \{\neg\psi, @_i\psi, \Box\psi\} \\ quan(\psi) + 1 & \text{if } \varphi = \forall p\psi \end{cases}$$

We prove the result by induction on pairs $(quan(\varphi), con(\varphi))$ ordered lexicographically, that is, $(q, c) < (q', c')$ iff (1) $q < q'$ or (2) $q = q'$ and $c < c'$.

Note that Lemma 13 has established this for all formulas φ associated with the pair $(0, 0)$ (that is, atomic formulas) and indeed for all formulas associated with pairs $(0, n)$ for any natural number n (that is, quantifier-free formulas). So the base cases are established, and as our inductive hypothesis (IH) we assume that for natural numbers q and c, with $q \leq c$, we have that $\mathcal{M}^\Gamma, g^\Gamma, u \models \varphi$ iff $\varphi \in u$ for all formulas associated with the pair (q, c)

So let θ be a formula associated with a pair with $c+1$ connectives. Now, if θ is a boolean, or of the form $\Box\psi$ or $@_i\psi$, we can argue as in Lemma 13, for all such formulas are associated with $(q, c+1)$, and we can use our new IH just as before. The critical case is when θ is of the form $\forall p\psi$. Note that such formulas are associated with $(q+1, c+1)$.

We want to show that $\mathcal{M}^\Gamma, g^\Gamma, u \vDash \varphi$ iff $\varphi \in u$ For the left-to-right direction, we show the contrapositive. So suppose that $\forall p\psi \notin u$. As u is an MCS, $\neg\forall p\psi \in u$, that is, $\exists p\neg\psi \in u$. Γ is witnessed, so for some proposition letter c, $@_j\neg\psi[c/p] \in \Gamma$. But then $\neg\psi[c/p] \in \Delta_j$. But $\neg\psi[c/p] \in \Delta_j$ is associated with $(q, c+1)$, so the IH applies and we have $\mathcal{M}^\Gamma, g^\Gamma, u \vDash \neg\psi[c/p]$. It follows from the Substitution Lemma that $\mathcal{M}^\Gamma, g', u \vDash \neg\psi$, that is, $\mathcal{M}^\Gamma, g', u \nvDash \psi$, where $g' \sim_p g^\Gamma$ is defined by setting $g'(p) = [\mathcal{M}^\Gamma, g^\Gamma]_c$. Hence $\mathcal{M}^\Gamma, g^\Gamma, u \nvDash \forall p\psi$, and we have proved the contrapositive.

For the right-to-left direction, suppose that $\forall p\psi \in u$, and further suppose for the sake of contradiction that $\mathcal{M}^\Gamma, g^\Gamma, u \nvDash \forall p\varphi$. Then for some $g' \sim_p g^\Gamma$ we have $\mathcal{M}^\Gamma, g', u \vDash \neg\varphi$. Now $g'(p) \in \Pi^\Gamma$, but by Lemma 14 we know $\Pi^\Gamma = [\mathcal{M}^\Gamma, g^\Gamma]^{sqf}$, hence $g'(p) = [\mathcal{M}^\Gamma, g^\Gamma]_\theta$ for some soft-QF formula θ. The Substitution Lemma tells us that $\mathcal{M}^\Gamma, g^\Gamma, u \vDash \neg\varphi[\theta/p]$ *iff* $\mathcal{M}^\Gamma, g', u \models \neg\varphi$, and hence we have $\mathcal{M}^\Gamma, g^\Gamma, u \vDash \neg\varphi[\theta/p]$. As $\neg\varphi[\theta/p]$ is associated with the pair $(q, c+1)$, so we can apply the IH to conclude $\neg\varphi[\theta/p] \in u$. But this leads to a contradiction. As u is an MCS, for all soft QF formulas θ, $\psi[\theta/p] \in u$ by the *Q2-sqf* axiom. In particular, $\varphi[\theta/p] \in u$. We conclude that $\mathcal{M}^\Gamma, g^\Gamma, u \vDash \forall p\varphi$ after all. □

Theorem 2 (Strong basic completeness). *Every* $\mathbf{K}_{SQph}$ + RULES-*consistent set of sentences has a named model.*

Proof. Follows from the previous lemma in the familiar way. □

So we have proved the *basic* strong completeness result. But the general model we have built is named, so we can immediately extend this to cover all pure extensions of $\mathbf{K}_{SQph}$ + RULES. In hybrid logic, a formula is called "pure" if all its atomic formulas are nominals. Here are three well-known examples: $i \to \Diamond i$ (the *Irreflexivity* axiom), $i \to \Box(\Diamond i \to i)$ (the *Antisymmetry* axiom), and $@_i\Diamond j \lor @_i j \lor @_j\Diamond i$ (the *Trichotomy* axiom). A pure formula φ *defines* a class of frames F iff: $(W, R) \in \mathsf{F} \Leftrightarrow (W, R) \models \varphi$. It is easy to check that our three examples define the class of ireflexive, antisymmetric, and trichotomous frames respectively. More importantly: adding *any* (combination of) pure formula(s) as extra axiom(s) to $\mathbf{K}_h$ + RULES is a proof system that is complete with respect to the class of frames defined. For a more detailed statement and discussion of this result, see [5]. Here we shall simply record that:

Theorem 3 (Strong pure completeness). *Let Λ be a set of pure formulas, and let* $\mathbf{K}_{SQph}$ + RULES + Λ *be the Hilbert-system obtained by using the pure formulas in Λ as extra axioms. Then every* $\mathbf{K}_{SQph}$ + RULES + Λ-*consistent set of sentences has a named model built over a Kripke frame belonging to the frame-class defined by the pure formulas in Λ.*

Proof. This is essentially the same as the proof of Theorem 7.29 from [6]. Because nominals directly "tag" worlds in the underlying Kripke frames, the standard completeness result for pure formulas carry over unchanged. □

Here is an example. Let $\Lambda = \{i \to \Diamond i, i \to \Box(\Diamond i \to i), \Diamond\Diamond i \to \Diamond i\}$. These three formulas jointly define the class of partially ordered frames. Adding them as axioms to $\mathbf{K}_{SQph}$ + RULES gives us the complete logic of this frame class.

6 Concluding remarks

In this paper we have extended completeness results for basic hybrid logic to cover languages containing propositional quantifiers. We adapted well-known techniques from the hybrid logic literature to build named general models, and thus prove completeness not merely for the minimal logic but for any pure extension. The key to this was our decision to directly "hardwire" nominals to worlds: this decoupled the world naming apparatus (nominals) from the quantificational apparatus (admissible propositions). Although we only treated the case for basic hybrid logic with a single modality, the results proved here can be extended to systems containing multiple modalities, the universal modality, the $\downarrow$-binder, and quantification over nominals, as we will show in an extended version of this paper. We also think the basic system outlined here hints at potentially useful applications. For example, Belardinelli *et al* [3] argue that (multimodal) epistemic logic augmented with propositional quantifiers is a useful knowledge representation tool. Our results for pure extensions suggest that adding a hybrid component might make them even more useful for such tasks.

But to close the paper, we return to the work of Arthur Prior that inspired it. Arthur Prior was a pioneer of propositional quantification in modal logic (see, in particular, [18,19]), and his students Robert Bull [9] and Kit Fine [11] both published technical results about it, the latter paper becoming highly influential. But Arthur Prior was also the inventor of hybrid logic, and in the final years of his career, these two interests became intertwined. Prior had oscillated between the "tag" view of nominals that is now standard and a "telescope" view that views them as (something more like) an infinite conjunction of information (see [7,14]). In two key late papers, Prior seems to have moved towards the "tag" view of nominals.[8] He also realised — anticipating the mantra of the Amsterdam school of modal logic — that modalities could be used to talk about absolutely *anything*. Indeed, his egocentric logic is an early example of what is now called description logic [2]. In egocentric logic, "worlds" are people and their properties and relationships (for example, their relative heights) are described using what Prior called "quasi modalities", with the help of propositional quantifiers and "people propositions" (nominals). Prior's death left many of these ideas underexplored, but it is clear that in his final years Prior developed several philosophically and technically novel systems, often involving both nominal-like entities and/or propositional quantification ("Prior's cocktail"). We want to use the language presented in this paper to explore this work more systematically.

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[8] Namely: "Tense logic and the logic of earlier and later", and "Quasi-propositions and quasi-individuals", both of which can be found in the first edition of *Papers on Time and Tense* [16]. The new edition [17] contains several more papers that build on these two, including "Egocentric logic". See Kofod's PhD thesis [14] for further discussion.

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